Problem 5.1

a. For $y = x^2$, we can parametrize w/ λ in $D=2$:

$$x^\mu \doteq \begin{pmatrix} \lambda \\ \lambda^2 \end{pmatrix}$$

then the curvature <u>vector</u> is: $K^\alpha = \frac{\ddot{x}^\alpha}{\dot{x}^\beta \dot{x}_\beta} - \frac{\dot{x}^\alpha (\ddot{x}^\gamma \dot{x}_\gamma)}{(\dot{x}^\beta \dot{x}_\beta)^2}$

w/ $\dot{x}^\alpha \doteq \begin{pmatrix} 1 \\ 2\lambda \end{pmatrix}$ & $\ddot{x}^\alpha \doteq \begin{pmatrix} \emptyset \\ 2 \end{pmatrix}$ so: $\dot{x}^\beta \dot{x}_\beta = (1+4\lambda^2)$

$$\ddot{x}^\gamma \dot{x}_\gamma = 4\lambda$$

&

$$K^\alpha \doteq \begin{pmatrix} \emptyset \\ \frac{2}{(1+4\lambda^2)} \end{pmatrix} - \frac{4\lambda}{(1+4\lambda^2)^2} \begin{pmatrix} 1 \\ 2\lambda \end{pmatrix}$$

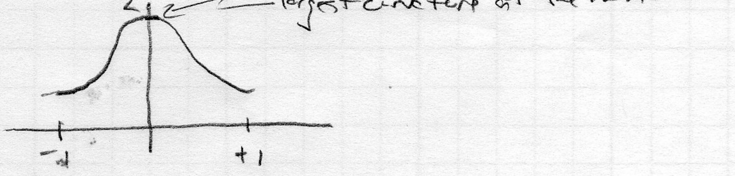
$$K^\alpha K_\alpha \doteq \left(\frac{4\lambda}{(1+4\lambda^2)^2}\right)^2 + \left[\frac{2}{(1+4\lambda^2)} - \frac{8\lambda^2}{(1+4\lambda^2)^2}\right]^2$$

$$= \frac{16\lambda^2}{(1+4\lambda^2)^4} + \frac{4}{(1+4\lambda^2)^2} - \frac{32\lambda^2}{(1+4\lambda^2)^3} + \frac{64\lambda^4}{(1+4\lambda^2)^4}$$

$$= \left[16\lambda^2 + 4(1+8\lambda^2+16\lambda^4) - 32\lambda^2(1+4\lambda^2) + 64\lambda^4\right](1+4\lambda^2)^{-4}$$

$$= \frac{16\lambda^2 + 4}{(1+4\lambda^2)^4} = \frac{4(1+4\lambda^2)}{(1+4\lambda^2)^4}$$

or $\boxed{K = \frac{2}{(1+4\lambda^2)^{3/2}}}$



b. For an ellipse parametrized by ϕ, $r(\phi) = \frac{p}{1+e\cos\phi}$

&

$$x^\mu(\phi) \doteq \begin{pmatrix} \frac{p}{1+e\cos\phi} \cdot \cos\phi \\ \frac{p}{1+e\cos\phi} \sin\phi \end{pmatrix} \quad \text{w/} \quad \dot{x}^\mu(\phi) = \frac{1}{(1+e\cos\phi)^2} \begin{pmatrix} -p\sin\phi \\ p(e+\cos\phi) \end{pmatrix}$$

$\ddot{x}^\mu(\phi)$

Problem 5.1 b

In[1]:=
```
r = p / (1 + e Cos[P]);
X = {r Cos[P], r Sin[P]};
```

In[3]:=
```
DX = D[X, P];
DDX = D[DX, P];
```

In[6]:=
```
k = Simplify[DDX / DX.DX - DX (DDX.DX) / (DX.DX) ^2]
```

Out[6]= $\left\{-\frac{(e+\text{Cos}[P])\,(1+e\,\text{Cos}[P])^3}{p\,\left(1+e^2+2\,e\,\text{Cos}[P]\right)^2},\ -\frac{(1+e\,\text{Cos}[P])^3\,\text{Sin}[P]}{p\,\left(1+e^2+2\,e\,\text{Cos}[P]\right)^2}\right\}$

In[7]:=
```
Simplify[k.k]
```

Out[7]= $\frac{(1+e\,\text{Cos}[P])^6}{p^2\,\left(1+e^2+2\,e\,\text{Cos}[P]\right)^3}$

In[8]:= $\text{kappa} = \frac{(1+e\,\text{Cos}[P])^3}{p\,\left(1+e^2+2\,e\,\text{Cos}[P]\right)^{3/2}}$

Out[8]= $\frac{(1+e\,\text{Cos}[P])^3}{p\,\left(1+e^2+2\,e\,\text{Cos}[P]\right)^{3/2}}$

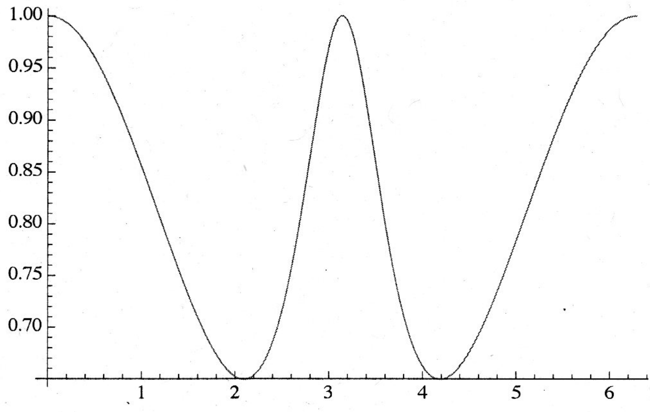
```
(* Check circular limit *)
```

In[10]:=
```
kappa /. e → 0
```

Out[10]= $\frac{1}{p}$

In[11]:=
```
Plot[(kappa /. {e → .5, p → 1}), {P, 0, 2 Pi}]
```

Out[11]=



Problem 5.2

a. $$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = A_{\nu,\mu} - A_{\mu,\nu}$$

the derivatives of $F_{\mu\nu}$: $F_{\mu\nu,\gamma} = A_{\nu,\mu\gamma} - A_{\mu,\nu\gamma}$

$$F_{\gamma\mu,\nu} = A_{\mu,\gamma\nu} - A_{\gamma,\mu\nu}$$

$$F_{\nu\gamma,\mu} = A_{\gamma,\nu\mu} - A_{\nu,\gamma\mu}$$

Then $F_{\mu\nu,\gamma} + F_{\gamma\mu,\nu} + F_{\nu\gamma,\mu} = \emptyset$.

b. Bianchi is: $R^\alpha{}_{\mu\nu\gamma;\rho} + R^\alpha{}_{\mu\rho\nu;\gamma} + R^\alpha{}_{\mu\gamma\rho;\nu} = \emptyset$.

If we sum by setting $\alpha = \nu$:

$$R^\alpha{}_{\mu\alpha\gamma;\rho} + R^\alpha{}_{\mu\rho\alpha;\gamma} + R^\alpha{}_{\mu\gamma\rho;\alpha} = \emptyset.$$

then $$R_{\mu\gamma;\rho} - R_{\mu\rho;\gamma} + R^\alpha{}_{\mu\gamma\rho;\alpha} = \emptyset$$

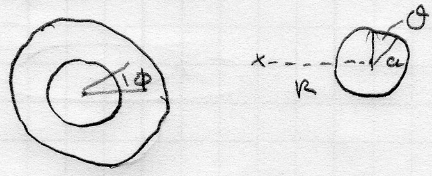
& hitting this w/ $g^{\mu\gamma}$ gives:

$$\underbrace{R^\gamma{}_{\gamma;\rho}} - R^\gamma{}_{\rho;\gamma} + R^{\alpha\gamma}{}_{\gamma\rho;\alpha} = \emptyset$$

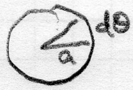
$$R_{;\rho} - R^\gamma{}_{\rho;\gamma} - R^\alpha{}_{\rho;\alpha} = \emptyset$$

or, finally $\boxed{R_{,\rho} = R_{;\rho} = 2R^\gamma{}_{\rho;\gamma}.}$

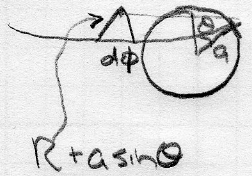
Problem 5.3



If we go a small $d\theta$, then $dl^2 = a^2 d\theta^2$



For a small $d\phi$



$dl^2 = (R+a\sin\theta)^2 d\phi^2$

putting these together, we have line element:

$$dl^2 = a^2 d\theta^2 + (R+a\sin\theta)^2 d\phi^2$$

& the metric, for $dx^\mu \doteq \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$ is

$$g_{\mu\nu} \doteq \begin{pmatrix} a^2 & 0 \\ 0 & (R+a\sin\theta)^2 \end{pmatrix} \to g^{\mu\nu} \doteq \begin{pmatrix} 1/a^2 & 0 \\ 0 & (R+a\sin\theta)^{-2} \end{pmatrix}$$

The structure of this metric is identical to that of a sphere, so we expect only $\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta}$ & $\Gamma^{\theta}_{\phi\phi}$ to be non-zero: (see (8.8) in Lecture 8)

$$\Gamma^{\phi}_{\theta\phi} = \frac{1}{2} g^{\phi\rho}\left(g_{\rho\theta,\phi} + g_{\rho\phi,\theta} - g_{\theta\phi,\rho}\right)$$

$$\overset{!}{=} \frac{1}{2} g^{\phi\phi} g_{\phi\phi,\theta} = \frac{1}{2}(R+a\sin\theta)^{-2} \cdot 2(R+a\sin\theta)\cdot a\cos\theta$$

$$\boxed{\Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} \overset{!}{=} \frac{a\cos\theta}{R+a\sin\theta}}$$

and $\Gamma^{\theta}_{\phi\phi} = \frac{1}{2} g^{\theta\rho}\left(g_{\rho\phi,\phi} + g_{\rho\phi,\phi} - g_{\phi\phi,\rho}\right) = -\frac{1}{2} g^{\theta\theta} g_{\phi\phi,\theta}$

$$\overset{!}{=} -\frac{1}{a^2}(R+a\sin\theta)\,a\cos\theta$$

or

$$\boxed{\Gamma^{\theta}_{\phi\phi} = -\frac{1}{a}(R+a\sin\theta)\cos\theta}$$

Problem 5.3 (continued)

$$R^{\alpha}{}_{\mu\nu\sigma} = -\Gamma^{\alpha}{}_{\mu\nu,\sigma} + \Gamma^{\alpha}{}_{\mu\sigma,\nu} + \Gamma^{\alpha}{}_{\nu\rho}\Gamma^{\rho}{}_{\mu\sigma} - \Gamma^{\alpha}{}_{\sigma\rho}\Gamma^{\rho}{}_{\mu\nu}$$

we get non-zero components only for $\nu \neq \sigma$, so:

$$R^{\alpha}{}_{\mu\theta\phi} = -\Gamma^{\alpha}{}_{\mu\theta,\phi} + \Gamma^{\alpha}{}_{\mu\phi,\theta} + \Gamma^{\alpha}{}_{\theta\rho}\Gamma^{\rho}{}_{\mu\phi} - \Gamma^{\alpha}{}_{\phi\rho}\Gamma^{\rho}{}_{\mu\theta}$$

— Take $\alpha = \theta$, then:

$$R^{\theta}{}_{\mu\theta\phi} = \Gamma^{\theta}{}_{\mu\phi,\theta} - \Gamma^{\theta}{}_{\phi\phi}\Gamma^{\phi}{}_{\mu\theta} \quad \& \ \mu = \phi \text{ contributes}$$

$$\Gamma^{\theta}{}_{\phi\phi,\theta} = \frac{1}{a}(R + a\sin\theta)\sin\theta - \frac{\cos\theta}{a}\, a\cos\theta$$

$$\overset{!}{=} \frac{R}{a}\sin\theta + (\sin^2\theta - \cos^2\theta).$$

$$\Gamma^{\theta}{}_{\phi\phi}\Gamma^{\phi}{}_{\phi\theta} = -\cos^2\theta$$

so $$\boxed{R^{\theta}{}_{\phi\theta\phi} = \frac{R}{a}\sin\theta + \sin^2\theta = -R^{\theta}{}_{\phi\phi\theta}}$$

— Set $\alpha = \phi$: $R^{\phi}{}_{\mu\theta\phi} = \Gamma^{\phi}{}_{\mu\phi,\theta} + \Gamma^{\phi}{}_{\theta\phi}\Gamma^{\phi}{}_{\mu\phi}$ & $\mu = \theta$ contributes

$$\Gamma^{\phi}{}_{\theta\phi,\theta} = \frac{-a\sin\theta}{R + a\sin\theta} - \frac{a\cos\theta}{(R + a\sin\theta)^2}\cdot(a\cos\theta)$$

$$\Gamma^{\phi}{}_{\theta\phi}\Gamma^{\phi}{}_{\theta\phi} = \frac{a^2\cos^2\theta}{(R + a\sin\theta)^2}$$

$$\boxed{-R^{\phi}{}_{\theta\phi\theta} = R^{\phi}{}_{\theta\theta\phi} = \frac{-a\sin\theta}{R + a\sin\theta}}$$

The Ricci tensor: $R^{\mu}{}_{\alpha\mu\beta} = R^{\theta}{}_{\alpha\theta\beta} + R^{\phi}{}_{\alpha\phi\beta}$

$$\boxed{R_{\alpha\beta} \doteq \begin{pmatrix} R^{\phi}{}_{\theta\phi\theta} & 0 \\ 0 & R^{\theta}{}_{\phi\theta\phi} \end{pmatrix} = \begin{pmatrix} \frac{a\sin\theta}{R + a\sin\theta} & 0 \\ 0 & \frac{R}{a}\sin\theta + \sin^2\theta \end{pmatrix}}$$

&

$$R = g^{\alpha\beta}R_{\alpha\beta} = \frac{1}{a^2}\frac{a\sin\theta}{R + a\sin\theta} + (R + a\sin\theta)^{-2}\left[\frac{1}{a}\sin\theta(R + a\sin\theta)\right]$$

$$\boxed{R \overset{!}{=} \frac{2\sin\theta}{a(R + a\sin\theta)}}$$

b. The symmetries we counted are for $R_{\alpha\beta\gamma\delta}$, & if we compute the two components of the Riemann tensor in the fully covariant form:

$$\left.\begin{aligned} R_{\theta\phi\theta\phi} &= g_{\theta\theta}R^{\theta}{}_{\phi\theta\phi} = a\sin\theta(R + a\sin\theta) \\ \& \ R_{\phi\theta\phi\theta} &= g_{\phi\phi}R^{\phi}{}_{\theta\phi\theta} = a\sin\theta(R + a\sin\theta) \end{aligned}\right\} \text{the same} \Rightarrow 1 \text{ indep. component to } R_{\alpha\mu\nu\gamma}.$$

Problem 5.4

A cylinder has metric given by the line element:

$$dl^2 = ds^2 + s^2 d\phi^2 + dz^2.$$

Its surface is two dimensional (no "ds" movement allowed):

$$dl^2 = s^2 d\phi^2 + dz^2.$$

where s is now a constant. Define the coordinate $y \equiv s\phi$, then $s^2 d\phi^2 = dy^2$ &

$$dl^2 = dy^2 + dz^2.$$

In this coordinate system, $g_{\mu\nu} \doteq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the space is flat, hence

$$\boxed{R^\alpha{}_{\mu\nu\beta} = \emptyset}$$

Problem 5.5

From $g_{\mu\nu,\gamma} = \emptyset$, we know only that $g_{\mu\nu}$ can be viewed as a constant, symmetric matrix:

$$g_{\mu\nu} \doteq \mathbb{G}$$

Take $dl^2 = dx^\mu g_{\mu\nu} dx^\nu \doteq dx^T \mathbb{G} dx$

Since $\mathbb{G}$ is sym. & invertible, it can be decomposed into:

$$\mathbb{G} = \mathbb{V}^T \Sigma \mathbb{V} \quad w/ \ \mathbb{V}^T\mathbb{V} = \mathbb{I} \quad (\mathbb{V} \text{ orthogonal})$$

and Σ diagonal w/ real, non-zero entries: $\{\sigma_i\}_{i=1}^{N}$

Then:

$$dl^2 = dx^T \mathbb{V}^T \Sigma \mathbb{V} dx \quad \text{let } y \equiv \overset{\text{constants}}{\mathbb{V}} x$$

$$= dy^T \Sigma dy$$

now we have a diagonal metric, & the line elt. takes the form:

$$dl^2 = \sum_{j=1}^{N} \sigma_j (dy^j)^2$$

let $z^j \equiv \sqrt{\sigma_j}\, dy^j$, we have $dl^2 = dz^j \delta_{ij} dz^i$

metric is identity.