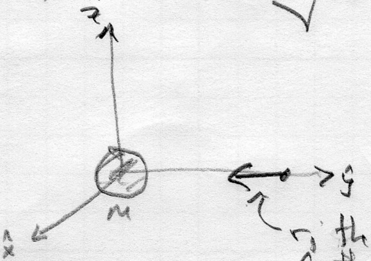
Problem 2.1

a. We cannot use $\rho(\phi)$ or $r(\phi)$ for purely radial motion, since in the case of, for example, motion along the $\hat{y}$ axis, $\phi = \pi/2$ is constant



r, the distance of the particle from the center of the central body cannot be parametrized by ϕ in this case — the body falls towards $r=0$ along a constant $\phi = \pi/2$

b. We must go back to the temporal (or related) parametrization:

$$L = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r).$$

then the Hamiltonian is:

$$H = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + U(r) = E,$$ the energy.

if we set $\phi = 0$, so the motion is along the $\hat{x}$ axis (say), then

$$E = \tfrac{1}{2}m\dot{r}^2 - \frac{GMm}{r}$$

we start from spatial infinity at rest ($\dot{r}(t=-\infty) = 0$, $r(t=-\infty) = \infty$)
so the energy for this trajectory is

$$E = 0 \quad (E \text{ is constant})$$

Then we need to solve:

$$\dot{r}^2 = \frac{2MG}{r} \Rightarrow \dot{r} = -\sqrt{\frac{2MG}{r}}$$

we are falling inward

to

$$\int_{r(0)}^{r(t)} dr\, r^{1/2} = -\int_0^t \sqrt{2MG}$$

$$\tfrac{2}{3}\left[r(t)^{3/2} - R^{3/2}\right] = -\sqrt{2MG}\, t$$

$$r(t)^{3/2} = R^{3/2} - 3\sqrt{\frac{MG}{2}}\, t$$

So

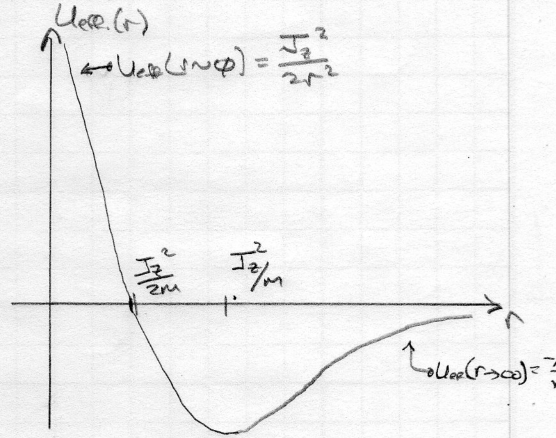
$$\boxed{r(t) = \left[R^{3/2} - 3\sqrt{\frac{MG}{2}}\, t\right]^{2/3}}$$

Problem 2.2

a. $L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$

$$H = \frac{\partial L}{\partial \dot{r}}\dot{r} + \frac{\partial L}{\partial \dot{\phi}}\dot{\phi} - L = (\dot{r}^2 + r^2\dot{\phi}^2) - \left[\tfrac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)\right]$$

$$= \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + U(r).$$



Using $\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \emptyset \Rightarrow r^2\dot{\phi} = J_z$, we have:

$$H = \tfrac{1}{2}\dot{r}^2 + \tfrac{1}{2}r^2 J_z^2/r^4 + U(r)$$

$$= \tfrac{1}{2}\dot{r}^2 + \left(\tfrac{1}{2}\frac{J_z^2}{r^2} - \frac{M}{r}\right)$$

and the effective potential energy is:

$$\boxed{U_{eff} = \frac{1}{2}\frac{J_z^2}{r^2} - \frac{M}{r}}$$

This has a zero at $U_{eff} = \emptyset \Rightarrow \frac{1}{2}\frac{J_z^2}{r^2} = \frac{M}{r} \Rightarrow \boxed{r_0 = \frac{J_z^2}{2M}}$

and a minimum at $\frac{dU_{eff}}{dr} = \emptyset \Rightarrow -\frac{J_z^2}{r^3} + \frac{M}{r^2} = \emptyset \Rightarrow J_z^2 = Mr \Rightarrow \boxed{r_{min} = J_z^2/M}$

b. If $E = U_{min}$: $U_{min} = U(r = J_z^2/M) = \frac{1}{2}\frac{M^2}{J_z^2} - \frac{M^2}{J_z^2} = -\frac{1}{2}\frac{M^2}{J_z^2}$

Then: $-\frac{1}{2}\frac{M^2}{J_z^2} = \frac{1}{2}\frac{J_z^2}{r^2} - \frac{M}{r} + \frac{1}{2}\dot{r}^2 \Rightarrow \dot{r}^2 = \emptyset$

and we have a quadratic in r: $\frac{1}{2}J_z^2 - Mr + \frac{1}{2}\frac{M^2}{J_z^2}r^2 = \emptyset$

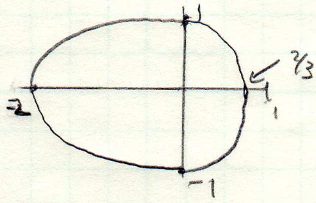
$$r = \frac{M \pm \sqrt{M^2 - \frac{4}{4}M^2}}{2\cdot\frac{1}{2}M^2/J_z^2} = J_z^2/M$$

the orbit is a circle (closest & furthest approach points are identical) w/ radius

$$\boxed{r = J_z^2/M}$$

Problem 2.3

a. For
$$r(\phi) = \frac{p}{1+e\cos\phi}$$

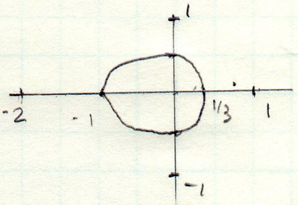


we have: $p=1$, $e=1/2$

ϕ	$r(\phi)$
$\emptyset$	$1/(1+1/2) = 2/3$
$\pi/2$	1
π	$1/(1-1/2) = 2$

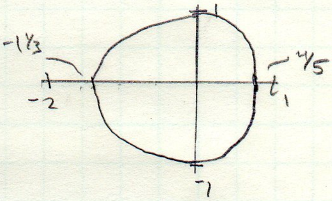
& $p=1/2$, $e=1/2$

ϕ	$r(\phi)$
$\emptyset$	$1/2/(1+1/2) = 1/3$
$\pi/2$	$1/2$
π	$1/2/(1-1/2) = 1$



$p=1$, $e=1/4$

ϕ	$r(\phi)$
$\emptyset$	$1/(1+1/4) = 4/5$
$\pi/2$	1
π	$1/(1-1/4) = 4/3$



A circle of radius R has $p = R, e = \emptyset$

b. Closest & furthest approach correspond to $\phi = \emptyset, \pi$ — so
$$r_p = \frac{p}{1+e} \qquad r_a = \frac{p}{1-e}$$

& then
$$p = (1+e)r_p \Rightarrow r_a = \frac{(1+e)}{(1-e)} r_p \Rightarrow r_a - e(r_a+r_p) = r_p$$

$$\boxed{e = \frac{r_a - r_p}{r_a + r_p}} \qquad \boxed{p = (1+e)r_p = \frac{2r_a r_p}{r_a+r_p}} \qquad (*)$$

c.
$$r(\phi) = \frac{1}{M/J_z^2 + \alpha\cos\phi} = \frac{J_z^2/M}{1+\frac{\alpha J_z^2}{M}\cos\phi}$$
so here, $p = J_z^2/M$, $e = \frac{\alpha J_z^2}{M}$

& using $(*)$

$$\alpha J_z^2/M = \frac{r_a - r_p}{r_a+r_p} \quad \& \quad J_z^2/M = \frac{2r_a r_p}{r_a + r_p} \Rightarrow \boxed{J_z = \pm\sqrt{\frac{2Mr_a r_p}{r_a+r_p}}}$$

$$\alpha\left(\frac{2r_a r_p}{r_a+r_p}\right) = \frac{r_a-r_p}{r_a+r_p} \Rightarrow \boxed{\alpha = \frac{r_a - r_p}{2r_a r_p}}$$

Problem 2.4

a. For the transformation: $x \longrightarrow x'$, the coordinate differential for x', viewing x' as a function of x is:

$$\boxed{dx'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} dx^{\beta}}$$ — transforms as a contravariant 1^{st} rank tensor.

A scalar $\Phi(x)$ responds to $x \rightarrow x'$ by transcription (i.e. not at all):

$$\Phi(x') = \Phi(x(x'))$$

the derivatives of $\Phi(x')$ w.r.t. x' (forming the gradient) can be written in terms of the derivatives of $\Phi(x)$ w.r.t. x:

$$\frac{\partial \Phi}{\partial x'^{\mu}} = \frac{\partial \Phi}{\partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial x'^{\mu}}$$

or, using $\Phi_{,\mu} \equiv \frac{\partial \Phi}{\partial x^{\mu}}$,

$$\boxed{\Phi'_{,\mu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \Phi_{,\alpha}}$$ — $\Phi_{,\mu}$ transforms as a covariant 1^{st} rank tensor.

b. For $x = s\cos\phi$, $dx = ds\cos\phi - s\sin\phi\, d\phi$
& $y = s\sin\phi$, $dy = ds\sin\phi + s\cos\phi\, d\phi$

The matrix equation relating those is

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\phi & -s\sin\phi \\ \sin\phi & s\cos\phi \end{pmatrix}}_{\mathbb{A}} \begin{pmatrix} ds \\ d\phi \end{pmatrix}$$

then inverting $\mathbb{A}$:

$$\begin{pmatrix} ds \\ d\phi \end{pmatrix} = \overbrace{\begin{pmatrix} \cos\phi & \sin\phi \\ -\frac{1}{s}\sin\phi & \frac{1}{s}\cos\phi \end{pmatrix}}^{\mathbb{A}^{-1}} \begin{pmatrix} dx \\ dy \end{pmatrix} \qquad (*)$$

From the coordinate differential transformation (contravariant 1^{st} rank tensor) we expect:

$$dx'^{\alpha} = \boxed{\frac{\partial x'^{\alpha}}{\partial x^{\beta}}} dx^{\beta}$$

is this object represented by $\mathbb{A}^{-1}$?

to form $\frac{\partial x'^{\alpha}}{\partial x^{\beta}}$, we need $x'^{\alpha}(x)$: $s = \sqrt{x^2+y^2}$ $\quad \phi = \tan^{-1}(y/x)$

then:
$$\frac{\partial s}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \quad \frac{\partial s}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \quad \frac{\partial \phi}{\partial x} = \frac{-y}{x^2+y^2} \quad \frac{\partial \phi}{\partial y} = \frac{x}{x^2+y^2}$$

to compare w/ $\mathbb{A}^{-1}$, we need to express: $\frac{\partial x'^{\alpha}}{\partial x^{\beta}} \doteq \begin{pmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix}$ in terms of s & ϕ

$$\boxed{\frac{\partial x'^{\alpha}}{\partial x^{\beta}} \doteq \begin{pmatrix} \cos\phi & \sin\phi \\ -\frac{\sin\phi}{s} & \frac{\cos\phi}{s} \end{pmatrix} = \mathbb{A}^{-1}}$$ ✓

Problem 2.4 (continued)

c. For $\Psi = kxy$, we have: $\Psi_{,\mu} \doteq \begin{pmatrix} \frac{\partial \Psi}{\partial x} \\ \frac{\partial \Psi}{\partial y} \end{pmatrix} = \begin{pmatrix} ky \\ kx \end{pmatrix}$

& $\Psi = ks^2 \cos\phi \sin\phi$ (using $x = s\cos\phi$, $y = s\sin\phi$)
w/ $dx'^{\alpha} \doteq \begin{pmatrix} ds \\ d\phi \end{pmatrix}$ has:

$$\Psi'_{,\mu} \doteq \begin{pmatrix} \frac{\partial \Psi}{\partial s} \\ \frac{\partial \Psi}{\partial \phi} \end{pmatrix} = \begin{pmatrix} 2ks\cos\phi\sin\phi \\ ks^2(-\sin^2\phi + \cos^2\phi) \end{pmatrix} \quad ($$

Now we will check that

$\Psi'_{,\mu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \Psi_{,\alpha}$ — we must express $\Psi_{,\mu}$ in term of s & ϕ:

$\Psi_{,\mu}(s,\phi) = \begin{pmatrix} ks\sin\phi \\ ks\cos\phi \end{pmatrix}$, then the RHS of the covariant transformati rule gives:

$$\frac{\partial x^{a}}{\partial x'^{\mu}} \Psi_{,\alpha} \doteq \begin{pmatrix} \frac{\partial x}{\partial s}\Psi_{,x} + \frac{\partial y}{\partial s}\Psi_{,y} \\ \frac{\partial x}{\partial \phi}\Psi_{,x} + \frac{\partial y}{\partial \phi}\Psi_{,y} \end{pmatrix} = \begin{pmatrix} \cos\phi\cdot(ks\sin\phi) + \sin\phi(k \\ -s\sin\phi(ks\sin\phi) + s\cos\phi(\end{pmatrix}$$

$$\doteq \begin{pmatrix} 2ks\cos\phi\sin\phi \\ ks^2(-\sin^2\phi + \cos^2\phi) \end{pmatrix}, \text{ identical to } (*)$$

Problem 2.5

For $x^1 = x$, $x^2 = y$ $\quad x'_1 = s \quad x'_2 = \phi$

w/ $x = s\cos\phi$, $y = s\sin\phi$, we have:
$s = \sqrt{x^2+y^2}$ $\phi = \tan^{-1}(y/x)$

$$\frac{\partial x^\alpha}{\partial x'^\beta} \doteq \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\phi & -s\sin\phi \\ \sin\phi & s\cos\phi \end{pmatrix}}_{\text{in terms of } x'^\alpha} = \underbrace{\begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & -y \\ \frac{y}{\sqrt{x^2+y^2}} & x \end{pmatrix}}_{\text{in terms of } x^\alpha}$$

$$\frac{\partial x'^\alpha}{\partial x^\beta} \doteq \begin{pmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix}}_{\text{in terms of } x^\alpha} = \underbrace{\begin{pmatrix} \cos\phi & \sin\phi \\ -\frac{\sin\phi}{s} & \frac{\cos\phi}{s} \end{pmatrix}}_{\text{in terms of } x'^\alpha}$$

Now, in the x^α representation:

$$\frac{\partial x^\alpha}{\partial x'^\beta} \cdot \frac{\partial x'^\beta}{\partial x^\gamma} \doteq \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & -y \\ \frac{y}{\sqrt{x^2+y^2}} & x \end{pmatrix} \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix} = \begin{pmatrix} 1 & \emptyset \\ \emptyset & 1 \end{pmatrix} \checkmark$$

In the x'^α coordinates:

$$\frac{\partial x^\alpha}{\partial x'^\beta} \frac{\partial x'^\beta}{\partial x^\gamma} \doteq \begin{pmatrix} \cos\phi & -s\sin\phi \\ \sin\phi & s\cos\phi \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\frac{\sin\phi}{s} & \frac{\cos\phi}{s} \end{pmatrix} = \begin{pmatrix} 1 & \emptyset \\ \emptyset & 1 \end{pmatrix} \checkmark$$

Problem 2.6

a. The 2nd rank contravariant tensor made from 2 first rank contravariant tensors:

$$T^{\mu\nu} = f^\mu h^\nu$$

transforms as:

$$T'^{\mu\nu} = f'^\mu h'^\nu = \frac{\partial x'^\mu}{\partial x^\alpha} f^\alpha \frac{\partial x'^\nu}{\partial x^\beta} h^\beta$$

$$\overset{!}{=} \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta} f^\alpha h^\beta$$

$$\boxed{T'^{\mu\nu} \overset{!}{=} \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta} T^{\alpha\beta}}$$

b. A second rank covariant tensor $T_{\mu\nu} = f_\mu h_\nu$ transforms via:

$$T'_{\mu\nu} = f'_\mu h'_\nu = \frac{\partial x^\alpha}{\partial x'^\mu} f_\alpha \frac{\partial x^\beta}{\partial x'^\nu} h_\beta$$

$$\boxed{T'_{\mu\nu} \overset{!}{=} \left(\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\right) T_{\alpha\beta}}$$

c. For a contravariant f^α & covariant h_β, we form:

$$\psi = f^\alpha h_\alpha$$

then

$$\psi' = f'^\alpha h'_\alpha = \frac{\partial x'^\alpha}{\partial x^\mu} f^\mu \frac{\partial x^\gamma}{\partial x'^\alpha} h_\gamma$$

$$\overset{!}{=} \underbrace{\frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x^\gamma}{\partial x'^\alpha}} f^\mu h_\gamma$$

$$\overset{!}{=} \delta^\gamma_\mu f^\mu h_\gamma = f^\gamma h_\gamma = \psi$$

so $\boxed{\psi' = \psi \text{, a scaler}}$

d. If $h_{\mu\nu}$ is a covariant, 2nd rank tensor, then:

$$h'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} h_{\alpha\beta}$$

& let $A^\alpha_\mu = \frac{\partial x^\alpha}{\partial x'^\mu}$

$$\overset{!}{=} A^\alpha{}_\mu A^\beta{}_\nu h_{\alpha\beta}$$

$$\overset{!}{=} A^\alpha{}_\mu h_{\alpha\beta} A^\beta{}_\nu$$

we can view this as matrix multiplication: $\mathbb{h}' = \mathbb{A}^T \mathbb{h} \mathbb{A}$

Then $\mathbb{h}^{-1\prime} = \mathbb{A}^{-1}\mathbb{h}^{-1}(\mathbb{A}^T)^{-1}$ & we know that $\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x'^\mu}{\partial x^\beta} = \delta^\alpha_\beta$

so that: $\mathbb{A}^{-1} \doteq \frac{\partial x'^\mu}{\partial x^\beta}$ & by assumption (definition), $\mathbb{h}^{-1} \equiv h^{\alpha\beta}$

$$\boxed{h'^{\mu\nu} = \underbrace{\frac{\partial x'^\mu}{\partial x^\alpha}}_{\text{"}\mathbb{A}^{-1}\text{"}} h^{\alpha\beta} \underbrace{\frac{\partial x'^\nu}{\partial x^\beta}}_{\text{"}(\mathbb{A}^T)^{-1}\text{"}}}$$

so the inverse transforms as contravariant 2nd rank tenso