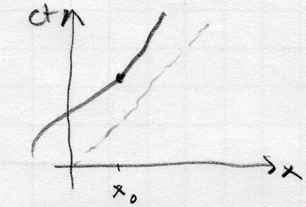
Problem 4.1



At x_0, the interval along the worldline of the particle, in x-t space, has:

$$ds^2 = -c^2dt^2 + dx^2$$

we want to find a boost $\Lambda^\mu{}_\alpha \ni$, in the new coordinate system, $-c^2d\bar{t}^2 = ds^2$ w/ $d\bar{x}^2 = \emptyset$, i.e.,

$$\begin{pmatrix} c\,d\bar{t} \\ d\bar{x} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} c\,dt \\ dx \end{pmatrix} \quad \text{w/ } d\bar{x} = \emptyset$$

so: $d\bar{x} = -\gamma\beta c\,dt + \gamma dx = \emptyset \rightarrow \beta = \frac{1}{c}\frac{dx}{dt}$ & since the particle is massive, we have, by assumption: $\frac{dx}{dt} = v < c$.

The boost is: $\boxed{\Lambda^\mu{}_\alpha \doteq \begin{pmatrix} +\gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \text{ w/ } \beta = \underbrace{\frac{dx}{dt}\Big|_{x=x_0}}_{v(x_0)} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}}$

Problem 4.2

Take an arbitrary linear combination of two connections:

$$T^\alpha{}_{\beta\gamma} = A\Gamma^\alpha{}_{\beta\gamma} + B\bar{\Gamma}^\alpha{}_{\beta\gamma}$$

then

$$T'^\alpha{}_{\beta\gamma} = A\Gamma'^\alpha{}_{\beta\gamma} + B\bar{\Gamma}'^\alpha{}_{\beta\gamma} = A\left[\frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial x'^\beta}\frac{\partial x^\rho}{\partial x'^\gamma}\Gamma^\mu{}_{\nu\rho} - \frac{\partial^2 x'^\alpha}{\partial x^\rho \partial x^\sigma}\frac{\partial x^\rho}{\partial x'^\beta}\frac{\partial x^\sigma}{\partial x'^\gamma}\right]$$

$$+ B\left[\frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial x'^\beta}\frac{\partial x^\rho}{\partial x'^\gamma}\bar{\Gamma}^\mu{}_{\nu\rho} - \frac{\partial^2 x'^\alpha}{\partial x^\rho \partial x^\sigma}\frac{\partial x^\rho}{\partial x'^\beta}\frac{\partial x^\sigma}{\partial x'^\gamma}\right]$$

$$= \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial x'^\beta}\frac{\partial x^\rho}{\partial x'^\gamma}T'^\mu{}_{\nu\rho} - (A+B)\frac{\partial^2 x'^\alpha}{\partial x^\rho \partial x^\sigma}\frac{\partial x^\rho}{\partial x'^\beta}\frac{\partial x^\sigma}{\partial x'^\gamma}$$

& $T^\mu{}_{\nu\rho}$ will be a tensor for $B = -A$:

$$\boxed{T^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma} - \bar{\Gamma}^\alpha{}_{\beta\gamma} \text{ is a tensor}}$$

Problem 4.3

a. For $\delta^\alpha_\beta \doteq \begin{cases} 1 & \alpha = \beta \\ \emptyset & \alpha \neq \beta \end{cases}$ we can transform (δ^α_β is a tensor, we're told)

$$\delta'^\alpha{}_\beta = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial x'^\beta}\delta^\mu{}_\nu = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x^\mu}{\partial x'^\beta} = \delta^\alpha{}_\beta$$

so this holds, numerically, in all coordinate systems.

b. The entries, assumed to be linearly independent, have:

$$\frac{\partial h_{\mu\nu}}{\partial h_{\alpha\beta}} = \delta^\alpha_\mu \delta^\beta_\nu \quad \text{i.e. only terms like } \frac{\partial h_{12}}{\partial h_{12}} = 1 \text{ are non-zero}$$

For $h^{\mu\nu} \doteq (h_{\mu\nu})^{-1}$, we have: $h^{\mu\gamma}h_{\gamma\nu} = \delta^\mu{}_\nu$

$$\frac{\partial}{\partial h_{\alpha\beta}}(h^{\mu\gamma}h_{\gamma\nu}) = \emptyset.$$

$$\Rightarrow \frac{\partial h^{\mu\gamma}}{\partial h_{\alpha\beta}}h_{\gamma\nu} + h^{\mu\gamma}\frac{\partial h_{\gamma\nu}}{\partial h_{\alpha\beta}} = \emptyset$$

or

$$\frac{\partial h^{\mu\gamma}}{\partial h_{\alpha\beta}}h_{\gamma\nu} = -h^{\mu\gamma}(\delta^\alpha_\gamma \delta^\beta_\nu)$$

$$\frac{\partial h^{\mu\gamma}}{\partial h_{\alpha\beta}}h_{\gamma\nu} = -h^{\mu\alpha}\delta^\beta_\nu$$

multiply both sides by $h^{\nu\sigma}$ to get rid of $h_{\gamma\nu}$ on the left.

$$\frac{\partial h^{\mu\gamma}}{\partial h_{\alpha\beta}}\delta^\sigma_\gamma = -h^{\mu\alpha}\delta^\beta_\nu h^{\nu\sigma}$$

$$\frac{\partial h^{\mu\sigma}}{\partial h_{\alpha\beta}} = -h^{\mu\alpha}h^{\beta\sigma}$$

so

$$\boxed{\frac{\partial h^{\mu\nu}}{\partial h_{\alpha\beta}} = -h^{\mu\alpha}h^{\beta\nu}}$$

Problem 4.4

a. We want to construct: f^α w/

$$\frac{df^\alpha}{d\gamma} + \Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta f^\gamma = \emptyset \quad \text{(definition of } \parallel\text{-transport)}$$

parametrize our curve via: $x^\mu(\Phi) \doteq \begin{pmatrix}\theta_0 \\ \Phi\end{pmatrix}$ for $\Phi = \emptyset \to 2\pi$. (i.e. $\gamma = \Phi$).

then $\dot{x}^\mu \doteq \begin{pmatrix} 0 \\ 1\end{pmatrix}$

For a sphere, we know: $\Gamma^\Phi{}_{\theta\Phi} = \Gamma^\Phi{}_{\Phi\theta} = \cot\theta$ & $\Gamma^\theta{}_{\Phi\Phi} = -\cos\theta\sin\theta$

Then the parallel-transport eqn. has 2 components $\alpha = \theta, \Phi$ for $f^\alpha \doteq \begin{pmatrix} f^\theta \\ f^\Phi\end{pmatrix}$

$\frac{df^\theta}{d\Phi}$ or $\to$ $\dot{f}^\theta + \Gamma^\theta{}_{\Phi\Phi}\dot{x}^\Phi f^\Phi = \emptyset$

$\frac{df^\Phi}{d\Phi}$ $\to$ $\dot{f}^\Phi + \Gamma^\Phi{}_{\theta\Phi}\dot{x}^\theta f^\Phi + \Gamma^\Phi{}_{\Phi\theta}\dot{x}^\Phi f^\theta = \emptyset$

or, explicitly, using the connection & $\dot{x}^\mu$

$$\dot{f}^\theta - \sin\theta_0\cos\theta_0 f^\Phi = \emptyset \qquad \dot{f}^\Phi + \cot\theta_0 f^\theta = \emptyset.$$

take the Φ-derivative of and input $\dot{f}^\theta$ from: $\ddot{f}^\Phi + \cot\theta_0\dot{f}^\theta = \emptyset$

$$\ddot{f}^\Phi + \cot\theta_0\sin\theta_0\cos\theta_0 f^\Phi = \emptyset \Rightarrow \ddot{f}^\Phi = -\cos^2\theta_0 f^\Phi$$

& $\boxed{f^\Phi = A\cos[\cos\theta_0\Phi] + B\sin[\cos\theta_0\Phi].}$

Now we have: $\dot{f}^\theta - \sin\theta_0\cos\theta_0\left[A\cos[\cos\theta_0\Phi] + B\sin[\cos\theta_0\Phi]\right] = \emptyset$

so

$$f^\theta(\Phi) = \sin\theta_0\cos\theta_0\left[\frac{A}{\cos\theta_0}\sin(\cos\theta_0\Phi) - \frac{B}{\cos\theta_0}\cos[\cos\theta_0\Phi]\right]$$

$$\boxed{f^\theta \doteq \sin\theta_0\left[A\sin(\cos\theta_0\Phi) - B\cos(\cos\theta_0\Phi)\right]}$$

And we have to set the initial conditions:

$$\begin{pmatrix} f^\theta(\emptyset) \\ f^\Phi(\emptyset)\end{pmatrix} = \begin{pmatrix} -\sin\theta_0 B \\ A\end{pmatrix} = \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \Rightarrow B = \frac{-\alpha}{\sin\theta_0},\ A = \beta$$

$$\boxed{\begin{aligned} f^\theta(\Phi) &\doteq \sin\theta_0\left[\beta\sin(\cos\theta_0\Phi) + \frac{\alpha}{\sin\theta_0}\cos(\cos\theta_0\Phi)\right] \\ f^\Phi(\Phi) &= \beta\cos(\cos\theta_0\Phi) - \frac{\alpha}{\sin\theta_0}\sin(\cos\theta_0\Phi)\end{aligned}} \qquad (*)$$

Problem 4.4 (continued)

The magnitude of f^α is: (using $g_{\mu\nu} \doteq \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta_0 \end{pmatrix}$)

$$f^2 = f^\alpha g_{\alpha\beta} f^\beta = (f^\theta)^2 + \sin^2\theta_0 (f^\phi)^2$$

$$= \sin^2\theta_0 \left[\beta^2 \sin^2(\cos\theta_0 \phi) + \frac{2\alpha\beta}{\sin\theta_0} \sin(\cos\theta_0 \phi)\cos(\cos\theta_0 \phi) + \frac{\alpha^2}{\sin^2\theta_0}\cos^2(\cos\theta_0 \phi)\right]$$

$$+ \sin^2\theta_0 \left[\beta^2 \cos^2(\cos\theta_0 \phi) - \frac{2\alpha\beta}{\sin\theta_0} \sin(\cos\theta_0 \phi)\cos(\cos\theta_0 \phi) + \frac{\alpha^2}{\sin^2\theta_0}\sin^2(\cos\theta_0 \phi)\right]$$

$$= \sin^2\theta_0 \left(\beta^2 + \frac{\alpha^2}{\sin^2\theta_0}\right)$$

or $\boxed{f^2 = \alpha^2 + \beta^2 \sin^2\theta_0}$, constant.

b. For initial vectors $p^\alpha \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $q^\alpha \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we have:

$$p^\alpha(\phi) \doteq \begin{pmatrix} \cos(\cos\theta_0 \phi) \\ -\frac{1}{\sin\theta_0}\sin(\cos\theta_0 \phi) \end{pmatrix} \qquad q^\alpha(\phi) \doteq \begin{pmatrix} \sin\theta_0 \sin(\cos\theta_0 \phi) \\ \cos(\cos\theta_0 \phi) \end{pmatrix}$$

&

$$p^\alpha g_{\alpha\beta} q^\beta = p^\theta q^\theta + \sin^2\theta_0 p^\phi q^\phi$$

$$= \sin\theta_0 \cos(\cos\theta_0 \phi)\sin(\cos\theta_0 \phi) - \sin\theta_0 \sin(\cos\theta_0 \phi)\cos(\cos\theta_0 \phi)$$

$$= \emptyset$$, they are always $\perp$.

c. For $q^\alpha(\emptyset) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we have $q^\alpha(\phi) = \begin{pmatrix} \sin\theta_0 \sin(\cos\theta_0 \cdot 2\pi) \\ \cos(\cos\theta_0 \cdot 2\pi) \end{pmatrix}$

&

$$q^\alpha(\emptyset) g_{\alpha\beta} q^\beta(\emptyset) = \sin^2\theta_0 = q^\alpha(2\pi) g_{\alpha\beta} q^\beta(2\pi)$$ (length is preserved).

&

$$q^\alpha(\emptyset) g_{\alpha\beta} q^\beta(2\pi) = \sin^2\theta_0 \cos(2\pi\cos(\theta_0))$$

so

$$\cos(\psi) = \frac{q^\alpha(\emptyset) g_{\alpha\beta} q^\beta(2\pi)}{\sqrt{q^\gamma(\emptyset) q_\gamma(\emptyset)}\sqrt{q^\rho(2\pi) q_\rho(2\pi)}} = \cos(2\pi\cos\theta_0)$$

$$\Rightarrow \boxed{\psi = 2\pi\cos\theta_0}$$

(same is true for p^α).

Problem 4.5

a. For $x'^\alpha = x^\alpha + \epsilon p_\beta f^{\alpha\beta}(x)$, we have:

$$\boxed{J = \tfrac{1}{2} p_\alpha p_\beta f^{\alpha\beta}}$$

check: $\dfrac{\partial J}{\partial p_\alpha} = p_\beta f^{\alpha\beta}$ so this is correct form for $x'^\alpha = x^\alpha + \epsilon \dfrac{\partial J}{\partial p_\alpha}$.

The transformed momenta are given by: $p'_\alpha = p_\alpha - \epsilon \dfrac{\partial J}{\partial x^\alpha}$
or

$$\boxed{p'_\alpha = p_\alpha - \tfrac{1}{2}\epsilon\, p_\mu p_\nu f^{\mu\nu}{}_{,\alpha}}$$

b. For $H = \frac{1}{2} p_\alpha g^{\alpha\beta} p_\beta$, $J = p_\alpha f^{\alpha\beta} p_\beta$, — take out the $1/2$ – that can get combined w/ ϵ.
we can construct:

$$[H, J] = \frac{\partial H}{\partial x^\mu}\frac{\partial J}{\partial p_\mu} - \frac{\partial H}{\partial p_\mu}\frac{\partial J}{\partial x^\mu}.$$

w/ element given by the derivatives:

$$\frac{\partial H}{\partial x^\mu} = \tfrac{1}{2} p_\alpha g^{\alpha\beta}{}_{,\mu} p_\beta \qquad \frac{\partial J}{\partial p_\mu} = 2 f^{\mu\beta} p_\beta$$

$$\frac{\partial H}{\partial p_\mu} = g^{\mu\beta} p_\beta \qquad \frac{\partial J}{\partial x^\mu} = p_\alpha f^{\alpha\beta}{}_{,\mu} p_\beta.$$

then

$$[H, J] = \tfrac{1}{2} p_\alpha g^{\alpha\beta}{}_{,\mu} p_\beta \cdot 2 f^{\mu\gamma} p_\gamma - g^{\mu\beta} p_\beta p_\alpha f^{\alpha\gamma}{}_{,\mu} p_\gamma$$

$$= p_\alpha p_\beta p_\gamma [g^{\alpha\beta}{}_{,\mu} f^{\mu\gamma} - g^{\mu\beta} f^{\alpha\gamma}{}_{,\mu}]$$

now, using $g^{\alpha\beta}{}_{;\mu} = g^{\alpha\beta}{}_{,\mu} + \Gamma^\alpha_{\sigma\mu} g^{\sigma\beta} + \Gamma^\beta_{\sigma\mu} g^{\alpha\sigma} = \emptyset$.
& $f^{\alpha\gamma}{}_{;\mu} = f^{\alpha\gamma}{}_{,\mu} + \Gamma^\alpha_{\sigma\mu} f^{\sigma\gamma} + \Gamma^\gamma_{\sigma\mu} f^{\alpha\sigma}$

we can input:

$$[H, J] = p_\alpha p_\beta p_\gamma [-(\Gamma^\alpha_{\sigma\mu} g^{\sigma\beta} + \Gamma^\beta_{\sigma\mu} g^{\alpha\sigma}) f^{\mu\gamma} - g^{\mu\beta}(f^{\alpha\gamma}{}_{;\mu} - \Gamma^\alpha_{\sigma\mu} f^{\sigma\gamma} - \Gamma^\gamma_{\sigma\mu} f^{\alpha\sigma})]$$

$$= p_\alpha p_\beta p_\gamma [g^{\sigma\beta}(-\Gamma^\alpha_{\sigma\mu} f^{\mu\gamma} + \Gamma^\alpha_{\mu\sigma} f^{\mu\gamma}) - \Gamma^\beta_{\sigma\mu} g^{\alpha\sigma} f^{\mu\gamma} + g^{\mu\beta}\Gamma^\gamma_{\sigma\mu} f^{\alpha\sigma} - f^{\alpha\gamma;\beta}$$

$$= p_\alpha p_\beta p_\gamma [-\Gamma^\beta_{\sigma\mu} g^{\alpha\sigma} f^{\mu\gamma} + g^{\mu\alpha}\Gamma^\beta_{\sigma\mu} f^{\gamma\sigma} - f^{\alpha\gamma;\beta}]$$

$$= p_\alpha p_\beta p_\gamma [-f^{\alpha\gamma;\beta}] \quad \& \quad [H, J] = \emptyset \Rightarrow \boxed{f_{(\alpha\gamma;\beta)} = \emptyset}$$

totally symmetric part survives.

c. For $f^{\mu\nu} = g^{\mu\nu}$, we have, trivially, $g_{\mu\nu;\alpha} = \emptyset$, so

$g_{(\mu\nu;\alpha)} = \emptyset$ & $g_{\mu\nu}$ is a Killing tensor.

The conserved quantity is $J = p_\alpha p_\beta g^{\alpha\beta} = 2H$, the Hamiltonian, so we know that the transformation corresponds to τ - translation

curve-parameter