Relativistic Mechanics

Lecture 9

Physics 411
Classical Mechanics II

September 17th, 2007

We have developed some tensor language to describe familiar physics – we reviewed orbital motion from the Lagrangian and Hamiltonian points of view, and learned how to write the equations of motion generically in terms of a metric and its derivatives (in the special combination $\Gamma^{\alpha}{}_{\mu\nu}$). All of the examples involved were in flat, three-dimensional space, and the most we have done is introduce curvilinear coordinates, i.e. there has been no fundamental alteration of Euclidean geometry.

Our next step might be to discuss dynamics, or at least, parametrized "motion" in a truly curved space, and for that, we would need a digression into what curvature really is. We will do that, of course, but for now, I'd like to introduce four-dimensional space-time, a slightly different alteration of geometry. The space-time of special relativity is the basic framework for general relativity, so looking at four-dimensional Minkowski geometry, while still flat, allows us to introduce some of the ideas that will be useful when we take away the metric entirely. This is also a good time to talk about the meaning of parameters in equations of motion when time itself is a coordinate (also true in GR). Finally, we can look at the Killing vectors associated with the non-dynamical metric, and discuss the role of infinitesimal generators once again, this time for special relativistic mechanics.

9.1 Minkowski Metric

A defining feature of special relativity, that there is a universal speed measured to have the same value in any frame (any "laboratory", moving w.r.t. another or not), is expressed mathematically as:

$$-c^2\,dt^2 + dx^2 + dy^2 + dz^2 = -c^2\,d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2, \qquad (9.1)$$

where $dx^\alpha \doteq (c\,dt, dx, dy, dz)$ is the coordinate differential in one frame, and $d\bar{x}^\alpha \doteq (c\,d\bar{t}, d\bar{x}, d\bar{y}, d\bar{z})$ is the coordinate differential in another frame. The two frames are related via a Lorentz transformation: $\Lambda^\alpha{}_\beta$. In fact, we can view the Lorentz transformation as defined by the above – it is the linear transformation that makes the above true – if we take $d\bar{x}^\mu = \Lambda^\mu{}_\nu\,dx^\nu$, then

$$d\bar{x}^\mu\,\eta_{\mu\nu}\,d\bar{x}^\nu = \Lambda^\mu{}_\alpha\,\Lambda^\nu{}_\beta\,\eta_{\mu\nu}\,dx^\alpha\,dx^\beta, \tag{9.2}$$

where we define the Minkowski metric for Cartesian coordinates as

$$\eta_{\mu\nu} \doteq \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{9.3}$$

The requirement of (9.1), written in matrix form, is $\Lambda^T\,\eta\,\Lambda = \eta$. This is very much analagous to $R^T\,I\,R = I$ for rotations (where the metric "matrix" is just the identity).

The interesting new element of the Lorentz transformations is the "boost", a mixing of temporal and spatial coordinates. We typically describe a frame $\bar{\mathcal{O}}$ moving at speed v relative to $\mathcal{O}$ along a shared $\hat{\mathbf{x}}$ axis as in Figure 9.1 – then the Lorentz boost takes the form:

$$\Lambda^\mu{}_\nu \doteq \begin{pmatrix} \gamma & -\gamma\,\beta & 0 & 0 \\ -\gamma\,\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \qquad \beta \equiv \frac{v}{c}. \tag{9.4}$$

We also have all the rotation structure from Euclidean space built in – consider a transformation defined by $\Lambda^\mu{}_\nu$ expressed in block form:

$$\Lambda^\mu{}_\nu \doteq \begin{pmatrix} 1 & 0 \\ 0 & \mathbb{R} \end{pmatrix}, \tag{9.5}$$

(the upper entry is 1×1, while the lower right is a 3×3 matrix) for any matrix $\mathbb{R}$ with $\mathbb{R}^T\,\mathbb{R} = \mathbb{I}$ – then we again preserve $dx^\alpha\,\eta_{\alpha\beta}\,dx^\beta = d\bar{x}^\alpha\,\eta_{\alpha\beta}\,d\bar{x}^\beta$.

9.2 Lagrangian

We have two basic problems to address in generating the (a) Lagrangian appropriate to dynamics in special relativity. The first is the form of the free particle Lagrangian, and the second is the parametrization of free particle motion.

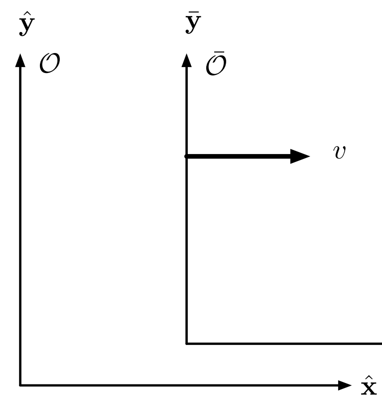

Figure 9.1: Coordinate systems related by a Lorentz boost – a frame $\mathcal{O}$ sees the $\bar{\mathcal{O}}$ moving to the right with speed v.

9.2.1 Length Extremization

To get the form, we can motivate from classical mechanics. For our usual free particle Lagrangian in three spatial dimensions with time playing the role of curve parameter, we have $L = \frac{1}{2} m v^2$. If we think of this in terms of the infinitesimal motion of a particle along a curve, we can write:

$$L = \frac{1}{2} m \frac{dx^\mu}{dt} g_{\mu\nu} \frac{dx^\nu}{dt} \tag{9.6}$$

with $g_{\mu\nu}$ the metric appropriate to the coordinates we are using. For concreteness, take $dx^\mu \doteq (dx, dy, dz)$, i.e. Cartesian coordinates – then $g_{\mu\nu}$ can be represented by the identity matrix. It is clear that the above is directly related to:

$$dr^2 = dx^\mu g_{\mu\nu} dx^\nu, \tag{9.7}$$

or what we would call the square of an infinitesimal distance, implicitly, "along the curve parametrized by t". If we make this explicit, associating $x^\mu(t)$ with a curve, then $dx^\mu = \frac{dx^\mu}{dt}$, and $dr^2 = \frac{dx^\mu}{dt} g_{\mu\nu} \frac{dx^\nu}{dt} dt^2$. Think of the action, $S = \int L dt$ – for L proportional to the length (squared) of the curve, we are, roughly speaking, minimizing the length of the curve itself (a.k.a. making a "line"). This is not entirely clear here, so we might ask what happens if we really take a Lagrangian proportional to dr – suppose we started with the action:

$$\boxed{S = \int dr = \int \sqrt{\dot{x}^\mu \, g_{\mu\nu} \, \dot{x}^\nu} dt.} \tag{9.8}$$

In two dimensions, which suffices, we have the integrand $L = \sqrt{\dot{x}^2 + \dot{y}^2}$, and the variation gives us:

$$0 = \frac{\dot{x}\,(\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x})}{(\dot{x}^2 + \dot{y}^2)^{3/2}} 0 \quad = \frac{\dot{y}\,(\dot{y}\,\ddot{x} - \dot{x}\,\ddot{y})}{(\dot{x}^2 + \dot{y}^2)^{3/2}}, \tag{9.9}$$

and these are degenerate equations – the solution is, for example

$$y(t) = A + B\,x(t), \tag{9.10}$$

an infinite family of extremal paths! Or is it? For any A and B, what we are really doing is writing y as a linear function of x, that defines a line as the curve in the $x-y$ plane, regardless of $x(t)$. We have, in effect, parametrized a line with x itself. That is a consequence of the manifest reparametrization invariance of this action, which is directly proportional to the length along the curve. We will see the same sort of action (by construction) for special relativity.

9.2.2 "Length" Extremization

On the relativistic side, we have a line element that is indefinite:

$$ds^2 = -c^2\,dt^2 + dx^2 + dy^2 + dz^2 = dx^\mu\,\eta_{\mu\nu}\,dx^\nu. \tag{9.11}$$

Now suppose we parametrize our curve in $D = 3+1$ via some ρ, to which we attach no physical significance. The goal of (special) relativistic mechanics is to find $x^\alpha(\rho)$, that is: $t(\rho)$, $x(\rho)$, etc. So we can once again write the ds^2 along the ρ-parametrized curve as:

$$ds^2 = \dot{x}^\mu\,\eta_{\mu\nu}\,\dot{x}^\nu\,d\rho^2 \qquad \dot{x}^\mu \equiv \frac{dx^\mu(\rho)}{d\rho} \tag{9.12}$$

Motivated by the above action (9.8) for a purely spatial geometry, take:

$$\boxed{S = \alpha \int ds = \alpha \int \sqrt{-\dot{x}^\mu\,\eta_{\mu\nu}\,\dot{x}^\nu}\,d\rho,} \tag{9.13}$$

[1] and call this the relativistic, free particle action. We again expect straight lines as the free particle motion, and we have gained reparametrization invariance as before. Explicitly, suppose we have in mind a function of ρ call it $\tau(\rho)$, and we want to rewrite this action in terms of τ – by the chain rule, we have

$$\dot{x}^\mu = \frac{dx^\mu(\tau(\rho))}{d\rho} = \frac{dx^\mu}{d\tau}\frac{d\tau}{d\rho}, \tag{9.14}$$

so that the action is

$$\begin{aligned} S &= \alpha \int \frac{d\tau}{d\rho}\sqrt{-\frac{dx^\mu}{d\tau}\,\eta_{\mu\nu}\,\frac{dx^\nu}{d\tau}}\,\frac{d\rho}{d\tau}d\tau \\ &= \alpha \int \sqrt{-\frac{dx^\mu}{d\tau}\,\eta_{\mu\nu}\,\frac{dx^\nu}{d\tau}}\,d\tau, \end{aligned} \tag{9.15}$$

and we have no way of establishing the difference between two different curve parametrizations from the action integrand alone. This gives us the freedom to *define* parametrizations of interest. In particular, we know that for any instantaneous velocity of a particle, it is possible to develop a Lorentz transformation to the local rest frame – that is, we can always generate a boost matrix that takes us from a "lab" in which a particle moves in time and space to the frame in which the particle moves only in time (use the instantaneous velocity to define the boost parameter γ – this is shown diagramatically in Figure 9.2).

This local rest frame, in which the particle moves only in a time-like direction serves to define the proper time τ of the particle. Since a generic trajectory will not have uniform velocity, τ changes along the trajectory, and we can define it only in an infinitesimal sense – we know there exists a "barred" frame (the local rest frame) in which a lab measurement of (dt, dx, dy, dz) is purely temporal:

$$-c^2\,dt^2 + dx^2 + dy^2 + dz^2 = -c^2\,d\bar{t}^2 \equiv -c^2\,d\tau^2. \tag{9.16}$$

Using this defining relation, we can write the derivative of τ w.r.t t and vice versa:

$$c^2\,dt^2\left[1 - \frac{1}{c^2}\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right)\right] = c^2\,d\tau^2 \tag{9.17}$$

[1] The minus sign under the square root is an artefact of the signature of our metric – for a particle at rest, moving through time only, we want a real action.

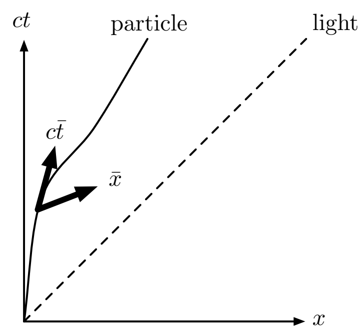

Figure 9.2: For a material particle trajectory in space time ($D = 1 + 1$ here), we can define a Lorentz boost to the frame $(c\,\bar{t}, \bar{x})$ shown such that the particle is instantaneously at rest (the $c\,\bar{t}$ axis is tangent to the particle trajectory) – physically, this is the frame in which the particle moves only in time, and defines the "proper time" of the particle.

where the term involving the squares of coordinate-time-derivatives is what we would call the instantaneous lab velocity of the particle. This provides a physical connection between the parameter τ and measurements made in the lab. We can write the above differential relation as

$$\dot{t}^2 = \frac{1}{1 - \frac{v^2}{c^2}}. \tag{9.18}$$

We started by suggesting that a natural Lagrangian for special relativity would be one that was proportional to the generalized length along the dynamical free particle curve. In order to see that this is a consistent description, we can take the relativistic Lagrangian (the integrand of the action) in coordinate time parametrization and see what it reduces to in the low-velocity limit where we should recover the classical free particle Lagrangian (the kinetic energy).

Since S is reparametrization invariant, it takes the same form for any parameter, in particular, for t parametrization:

$$S = \alpha \int \sqrt{-\frac{dx^\mu}{dt}\,\eta_{\mu\nu}\,\frac{dx^\nu}{dt}}\,dt \tag{9.19}$$

and we would call the Lagrangian (the integrand of the action)

$$L = \alpha\,\sqrt{c^2 - v^2} = \alpha\,c\,\sqrt{1 - \frac{v^2}{c^2}} \sim \alpha\,c - \frac{1}{2}\,\alpha\,\frac{v^2}{c}. \tag{9.20}$$

If we demand that this be the kinetic energy in the low velocity limit, then we must set $\alpha = -m\,c$. We then have an approximate Lagrangian that differs from $T = \frac{1}{2}\,m\,v^2$ only by a constant, and that will not change the equations of motion, it is ignorable under variation.

This leaves us with the final form for a free particle relativistic action and associated Lagrangian:

$$S = -m\,c \int \sqrt{-\frac{dx^\mu}{d\tau}\,\eta_{\mu\nu}\,\frac{dx^\nu}{d\tau}}\,d\tau \tag{9.21}$$

where we use the proper time as the parameter, and we can, at any point, connect this to coordinate time via (9.18). The advantage to proper time as a parameter is that the action is *manifestly* a scalar. We know that the four velocity $\frac{dx^\mu}{d\tau}$ is a contravariant four-vector (since dx^μ is, and $d\tau$ is clearly a scalar), the metric $\eta_{\mu\nu}$ is a covariant second rank tensor, so the term inside the square root is clearly a scalar.

Now we can begin the same sorts of analysis we did for the classical Lagrangian, using the above action as our starting point. In particular, it will be interesting to find the canonical infinitesimal generators associated with constants of the motion, although we already know basically what these are (Lorentz transformations, after all, have $\Lambda^T\,\eta\,\Lambda = \eta$).