Parametrized Motion

Lecture 8

Physics 411
Classical Mechanics II

September 14th 2007

We have our fancy new derivative, but what to do with it? In particular, how can we interpret "motion", the usual information of the derivative, in terms of this new construction? We will see, using the simplest possible curved space, that the connection provides the connection (so to speak). The new element in derivatives is the notion that the basis vectors themselves depend on position. This is not so surprising, we have seen that in the past (spherical, cylindrical coordinates with their position-dependent basis vectors) – what we will make explicit now is the role of $\Gamma^{\alpha}{}_{\beta\gamma}$ in keeping track of this change. The derivative of a vector field along a curve is made up of two parts: Changes to the vector field itself, and changes in the basis vectors.

8.1 Parallel Transport

Consider a vector $f^{\alpha}(x)$ defined on the surface of a sphere – the coordinates there are defined to be θ, ϕ. Suppose that we have in mind a definite curve of some sort, parametrized by τ – a curve defined by $(\theta(\tau), \phi(\tau)) \equiv x^{\alpha}(\tau)$.

From our three-dimensional point of view, the tangent to the curve is given by $\dot{x}(\tau)$. If we have a vector field $f^{\alpha}(x)$ defined on the sphere, then we can define $f^{\alpha}(\tau)$ using the curve definition. For concreteness, let

$$\tilde{\mathbf{f}} = \tilde{f}^{\theta}(\theta, \phi)\, \hat{\mathbf{e}}_{\theta} + \tilde{f}^{\phi}(\theta, \phi)\, \hat{\mathbf{e}}_{\phi} \tag{8.1}$$

in the natural (see Griffiths) orthonormal basis defined on the surface of the sphere. But we are not used to this basis, we are used to our so-called coordinate basis, which is not normalized. This is only a point of notation, but the idea of switching to the coordinate basis (and/or from it to the

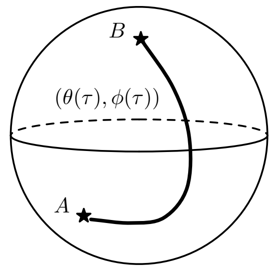

Figure 8.1: Going from point A to point B along a curve parametrized by τ defined on the surface of a sphere.

orthonormal basis) comes up a lot, depending on context. To change basis, recall our relation from last time:

$$\hat{\mathbf{e}}_{\mathbf{r}} = \mathbf{e}_{\mathbf{r}} \qquad \hat{\mathbf{e}}_\theta = \frac{1}{r}\,\mathbf{e}_\theta \qquad \hat{\mathbf{e}}_\phi = \frac{1}{r\,\sin\theta}\,\mathbf{e}_\phi \tag{8.2}$$

at which point the vector can be written

$$\begin{aligned}\tilde{\mathbf{f}} &= \frac{1}{r}\,\tilde{f}^\theta(\theta,\phi)\,\mathbf{e}_\theta + \tilde{f}^\phi(\theta,\phi)\,\frac{\mathbf{e}_\phi}{r\,\sin\theta} \\ &\equiv f^\theta\,\mathbf{e}_\theta + f^\phi\,\mathbf{e}_\phi,\end{aligned} \tag{8.3}$$

with $f^\theta = \frac{1}{r}\,\tilde{f}^\theta$ and $f^\phi = \tilde{f}^\phi\,r^{-1}\,\sin^{-1}\theta$. The above is what we normally have in mind when we write f^α.

And now we ask: if we go along the curve a distance $d\tau$, how does the vector f^α change?

$$\begin{aligned}d\mathbf{f}|_C = f^\theta_{,\alpha}\,\dot{x}^\alpha\,d\tau\,\mathbf{e}_\theta + f^\theta\left(\frac{\partial\mathbf{e}_\theta}{\partial\theta}\,\dot{\theta} + \frac{\partial\mathbf{e}_\theta}{\partial\phi}\,\dot{\phi}\right) \\ + f^\phi_{,\alpha}\,\dot{x}^\alpha\,d\tau\,\mathbf{e}_\phi + f^\phi\left(\frac{\partial\mathbf{e}_\phi}{\partial\theta}\,\dot{\theta} + \frac{\partial\mathbf{e}_\phi}{\partial\phi}\,\dot{\phi}\right).\end{aligned} \tag{8.4}$$

To evaluate the above, we need the derivatives of the basis vectors – they are (using our inversion formula or other)

$$\begin{aligned}&\frac{\partial\mathbf{e}_\theta}{\partial\theta} = -r\,\mathbf{e}_{\mathbf{r}} \qquad \frac{\partial\mathbf{e}_\theta}{\partial\phi} = \cot\theta\,\mathbf{e}_\phi \\ &\frac{\partial\mathbf{e}_\phi}{\partial\theta} = \cot\theta\,\mathbf{e}_\phi \qquad \frac{\partial\mathbf{e}_\phi}{\partial\phi} = -\sin\theta\,(\cos\theta\,\mathbf{e}_\theta + \sin\theta\,\mathbf{e}_{\mathbf{r}}).\end{aligned} \tag{8.5}$$

If we are on the surface of the sphere, there is no such thing as $\mathbf{e_r}$ – while it exists in three dimensions, there is no normal to the sphere in two. So restricting our attention, we have

$$\begin{aligned} d\mathbf{f}|_C &= \left(f^\theta_{,\alpha}\,\dot{x}^\alpha - f^\phi\,\sin\theta\,\cos\phi\,\dot{\phi} \right)\,\mathbf{e}_\theta\,d\tau \\ &+ \left(f^\phi_{,\alpha}\,\dot{x}^\alpha + \cot\theta\,(f^\theta\,\dot{\phi} + f^\phi\,\dot{\theta}) \right)\,\mathbf{e}_\phi\,d\tau. \end{aligned} \tag{8.6}$$

This could be written

$$d\mathbf{f}|_C \doteq \begin{pmatrix} f^\theta_{,\alpha}\,\dot{x}^\alpha \\ f^\phi_{,\alpha}\,\dot{x}^\alpha \end{pmatrix} d\tau + \begin{pmatrix} -\sin\theta\,\cos\theta\,f^\phi\,\dot{\phi} \\ \cot\theta\,(f^\theta\,\dot{\phi} + f^\phi\,\dot{\theta}) \end{pmatrix} d\tau, \tag{8.7}$$

and the second term has the form $C^\alpha{}_{\beta\gamma}\,\dot{x}^\beta\,f^\gamma$ if we define

$$C^\theta{}_{\phi\phi} = -\cos\theta\,\sin\theta \quad C^\phi{}_{\theta\phi} = C^\phi{}_{\phi\theta} = \cot\theta \quad \text{(all other components are zero)}. \tag{8.8}$$

Then we can express the above as:

$$\boxed{df^\alpha|_C = \left(f^\alpha{}_{,\gamma}\,\dot{x}^\gamma + C^\alpha{}_{\beta\gamma}\,\dot{x}^\beta\,f^\gamma \right)\,d\tau = \dot{x}^\gamma\,f^\alpha{}_{;\gamma}\,d\tau,} \tag{8.9}$$

so we see that we might well call $\frac{df^\alpha}{d\tau}|_C$ "the derivative along the curve parametrized by τ". This object is denoted

$$\boxed{\left.\frac{df^\alpha}{d\tau}\right|_C = \dot{x}^\gamma\,f^\alpha{}_{;\gamma} \equiv \frac{Df^\alpha}{D\tau}.} \tag{8.10}$$

Let's take a breather – what have we done? Well, I have shown that tensors have a notion of distance along a curve – the complication that leads to the appearance of $C^\alpha{}_{\beta\gamma}$ can either be viewed as the changing of basis vectors, or the lack of subtraction except at a point. These two views can both be used to describe the covariant derivative. In one case, we are explicitly inserting the basis vectors and making sure to take the derivative w.r.t. both the elements of f^α and the basis vectors. In the other, we are using the "fudge factor" to pick up the slack in the non-tensorial (ordinary) derivative.

Either way you like, we have a new notion of the change of a vector along a curve. Incidentally, in Cartesian coordinates, the equivalent object would look like $f^\alpha{}_{,\gamma}\,\dot{x}^\gamma$, and this highlights one common procedure we will encounter in a lot of GR, that of "minimal substitution". General relativity is a coordinate-independent theory, the best way to make true statements, then, is to make tensor statements. There is a lack of uniqueness that plagues the "non-tensor statements go to tensor statements" algorithm, and minimal replacement is (sometimes) a way around this. It certainly has worked out here – we take the non-tensor $f^\alpha_{\ ,\gamma}$ and replace it with the tensor $f^\alpha_{\ ;\gamma}$ to get the tensor form of "$\dot{f}^\alpha$".

But wait a minute – I haven't *proved* anything here, just suggested that we box up the terms that aren't familiar from (8.7) and defined a connection that has the correct form. I am entirely within my rights to do this, but I must show you that my construction of the connection has the appropriate transformation properties. I will bypass this entirely! You will see soon enough that the above definition (8.8) is in fact a connection, and a natural one given the surface of a sphere ("natural"?!).

Now, as with almost all studies of new derivative operators, we ask the very important question – what is a constant? In Cartesian coordinates, a vector field is constant if its components are . . . constant. In the general case, we must demand constancy w.r.t. the covariant derivative, i.e. we take a vector $f^\alpha(x)$ and require

$$\frac{Df^\alpha}{D\tau} = \dot{x}^\gamma \, f^\alpha_{\ ;\gamma} = 0. \tag{8.11}$$

Depending on our space, we can have a "constant" vector f^α that does not have constant entries – after all

$$\dot{x}^\gamma \, f^\alpha_{\ ;\gamma} = \dot{x}^\gamma \, f^\alpha_{\ ,\gamma} + C^\alpha_{\ \beta\gamma} \, \dot{x}^\gamma \, f^\beta = 0 \rightarrow \frac{df^\alpha}{d\tau} = -C^\alpha_{\ \beta\gamma} \, \dot{x}^\gamma \, f^\beta. \tag{8.12}$$

A vector satisfying this constancy condition is said to be "parallel transported around the curve defined by $x^\alpha(\tau)$ with tangent vector $\dot{x}^\alpha(\tau)$". Remember, there is a curve lurking in the background, otherwise, we have no τ.

The flip side of this discussion is that we can *make* a vector constant – using the above equation, we have a first order ODE for $f^\alpha(x)$ – we could solve it given the values of $f^\alpha(P)$ at a point $P = x(0)$. This allows us to explicitly *construct* constant vectors.

What properties do we expect for parallel transport? From its name, and the idea that the vectors f^α with $\frac{Df^\alpha}{D\tau} = 0$ are somehow "constant", we impose the condition that two vectors that are being parallel transported around the same curve C remain at the same angle w.r.t. each other.

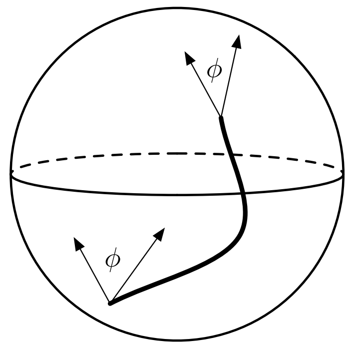

Figure 8.2: "angle conservation" under parallel transport of two vectors.

Same angle w.r.t. each other? We haven't even defined the angle between vectors, but it is what you expect – the length of a vector is given by $f^2 = f^\alpha g_{\alpha\beta} f^\beta$, and the angle between two vectors is similarly defined to be:

$$\cos\phi = \frac{\sqrt{p^\gamma\, q_\gamma}}{\sqrt{p^\alpha\, p_\alpha}\sqrt{q^\beta\, q_\beta}}. \tag{8.13}$$

So take two vectors parallel-transported around a curve C, we have $\frac{Dp^\alpha}{D\tau} = \frac{Dq^\alpha}{D\tau} = 0$, then the requirement that the angle remain constant along the curve is summed up in the following:

$$\dot{x}^\gamma \left(g_{\alpha\beta}\, p^\alpha\, q^\beta\right)_{;\gamma} = 0. \tag{8.14}$$

The covariant derivative satisfies all the usual product rules, so we can expand out the left-hand side:

$$\dot{x}^\gamma \left(g_{\alpha\beta;\gamma}\, p^\alpha\, q^\beta + g_{\alpha\beta}\, p^\alpha_{\ ;\gamma}\, q^\beta + g_{\alpha\beta}\, p^\alpha\, q^\beta_{\ ;\gamma}\right) = 0 \tag{8.15}$$

but the two terms involving the derivatives of p^α and q^α are zero by assumption, so we must have $\dot{x}^\gamma\, g_{\alpha\beta;\gamma} = 0$. If this is to be true for any curve $x^\alpha(\tau)$ and vectors p^α, q^α, then we must have $g_{\alpha\beta;\gamma} = 0$.

Well, as you will see in your homework, this leads to an interesting requirement:

$$\boxed{g_{\alpha\beta;\gamma} = 0 \rightarrow C^\mu_{\ \alpha\beta} = \frac{1}{2}\, g^{\mu\gamma}\left(g_{\gamma\beta,\alpha} + g_{\gamma\alpha,\beta} - g_{\alpha\beta,\gamma}\right) \equiv \Gamma^\mu_{\ \alpha\beta}.} \tag{8.16}$$

That's precisely what we defined to be $\Gamma^\mu_{\ \alpha\beta}$ during our discussion of Keplerian orbits.

8.2 Geodesics

In fact, the whole story becomes eerily similar to our second meeting – there is a special class of curves, ones whose tangent vector is parallel-transported along the curve – that is

$$\dot{x}^{\alpha}{}_{;\gamma}\,\dot{x}^{\gamma} = 0. \tag{8.17}$$

We will have more to say about this special class of curve later, but they do have the property of extremizing distance between points, that is, they are in a generalized sense, "straight lines".

Let me write out the requirement explicitly:

$$\dot{x}^{\alpha}{}_{;\gamma}\,\dot{x}^{\gamma} = \frac{d\dot{x}^{\alpha}}{d\tau} + \Gamma^{\alpha}{}_{\beta\gamma}\,\dot{x}^{\beta}\,\dot{x}^{\gamma} = 0 \tag{8.18}$$

which we sometimes write as

$$\ddot{x}^{\nu}\,g_{\alpha\nu} + \Gamma_{\alpha\beta\gamma}\,\dot{x}^{\beta}\,\dot{x}^{\gamma} = 0 \tag{8.19}$$

In three dimensional space with Cartesian coordinates, this reduces to $\ddot{x}^{\mu} = 0$, or the familiar lines of force-free motion: $x^{\alpha}(t) = A^{\alpha}\,t + B^{\alpha}$ for constants A^{α} and B^{α}. The connection coefficients for a constant metric have no derivatives, hence the simplification. We can use a non-trivial coordinate system (i.e. one with a non-vanishing connection) – take cylindrical coordinates: $x^{\mu} \doteq (s, \phi, z)$ with metric and connection:

$$g_{\mu\nu} \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & s^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Gamma^{s}{}_{\phi\phi} = -s \qquad \Gamma^{\phi}{}_{s\phi} = \Gamma^{\phi}{}_{\phi s} = \frac{1}{s}, \tag{8.20}$$

with all other elements of $\Gamma^{\alpha}{}_{\beta\gamma} = 0$. Then the equations for geodesic motion coming from (8.18) are:

$$\begin{aligned} \ddot{s} &= s\,\dot{\phi}^2 \\ \ddot{\phi} &= -\frac{2\,\dot{\phi}\,\dot{s}}{s} \\ \ddot{z} &= 0. \end{aligned} \tag{8.21}$$

The line solutions to this are less obvious than in the Cartesian case, $z(t) = A\,t + B$ is obvious, and decoupled from the planar portion. As a quick check, it is clear that $\phi = \phi_0$ (a constant angle) leads to $\ddot{s} = 0$, so straight line "radial" motion. To find the most general solution, we can take the

Cartesian case: $x(t) = F\,t + B$, $y(t) = P\,t + Q$ and construct $s = \sqrt{x^2 + y^2}$ and $\phi = \tan^{-1}(y/x)$, leading to:

$$s(t) = \sqrt{(F\,t + G)^2 + (P\,t + Q)^2} \quad \phi(t) = \tan^{-1}\left(\frac{P\,t + Q}{F\,t + G}\right), \tag{8.22}$$

which, combined with $z(t) = A\,t + B$ does indeed solve (8.21).