Hamiltonian Solution I

Lecture 5

Physics 411
Classical Mechanics II

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Here we continue with the Hamiltonian formulation of the central body problem – we will uncover the real power of the approach by considering transformations, finding conserved quantities and using them to reduce the number (and degree) of ODEs we get in the equations of motion.

Our first goal is to prove Noether's theorem on the Hamiltonian side, and we are poised to do this. Then we will develop constants of the motion for Euclidean space written in spherical coordinates. These, of course, correspond to angular momentum conservation and total energy conservation. We will focus on the solution of the ODE's that naturally arise in this situation – this will also shed some light on the equatorial plane issue discussed previously.

5.1 Canonical Infinitesimal Transformations

We had, at the end of last time, the generic form for a canonical transformation – one that led to a new Hamiltonian system (by which I mean one whose equations of motion are of the usual form). The transformation is generated by a function which we called $\bar{K}(x, P)$, and connects the "old" coordinates and momenta (x^α, p_α) with the new set (X^α, P_α) via:

$$\boxed{p_\alpha = \frac{\partial \bar{K}}{\partial x^\alpha} \qquad X^\alpha = \frac{\partial \bar{K}}{\partial P_\alpha}.} \tag{5.1}$$

The advantage of this form over the $K(x, X)$ form is clear from the identity transformation:

$$\bar{K} = x^\alpha P_\alpha. \tag{5.2}$$

This generates $X^\alpha = x^\alpha$ and $P_\alpha = p_\alpha$ trivially. We will be looking at small perturbations from identity, and the $\bar{K}(x, P)$ generator is well-suited because it is easy to write the identity transformation from this form.

Our goal here is to look at canonical *infinitesimal* transformations – the question will eventually be, what are the transformations that leave the entire problem invariant (and not just form invariant)? So we must first be able to talk about generic transformations.

Of course, most transformations are complicated, so in order to discuss a generic one, we make a small linear transformation. This turns out (for deep and not so deep reasons) to suffice – we can put together multiple infinitesimal transformations to make big complicated ones.

To start off, then, we make a small change to the coordinates and momenta:

$$\begin{aligned} X^\alpha &= x^\alpha + \epsilon\, f^\alpha(x, p) \\ P_\alpha &= p_\alpha + \epsilon\, h_\alpha(x, p). \end{aligned} \tag{5.3}$$

That is: our new coordinates and momenta differ from the old by a small amount and depend on functions of the old coordinates and momenta.

We must express this situation in terms of $\bar{K}$ – we want $\bar{K}$ to generate the small transformations above, so we add a little piece to the identity:

$$\bar{K} = x^\alpha\, P_\alpha + \epsilon\, J(x, P). \tag{5.4}$$

Well then, we have directly from the generator $\bar{K}$:

$$\begin{aligned} p_\alpha &= \frac{\partial \bar{K}}{\partial x^\alpha} = P_\alpha + \epsilon\, \frac{\partial J}{\partial x^\alpha} \\ X^\alpha &= \frac{\partial \bar{K}}{\partial P_\alpha} = x^\alpha + \epsilon\, \frac{\partial J}{\partial P_\alpha} = x^\alpha + \epsilon \left(\frac{\partial J}{\partial p_\beta}\, \frac{\partial p_\beta}{\partial P_\alpha} \right) = x^\alpha + \epsilon \left(\frac{\partial J}{\partial p_\beta} \delta^\alpha_\beta + O(\epsilon) \right) \end{aligned} \tag{5.5}$$

and comparing with (5.3), this tells us

$$\boxed{\begin{aligned} f^\alpha &= \frac{\partial J}{\partial p_\alpha} \\ h_\alpha &= -\frac{\partial J}{\partial x^\alpha}. \end{aligned}} \tag{5.6}$$

5.2 Rewriting H

Excellent – now what? Well, we have a transformation, generated by $J(x,p)$ (an infinitesimal generator) that is defined entirely in terms of the original coordinates and momenta. We can now ask how H changes under the transformation:

$$\begin{aligned} H(x+\epsilon\, f, p+\epsilon\, h) - H &= \epsilon \left(\frac{\partial H}{\partial x^\alpha} f^\alpha + \frac{\partial H}{\partial p_\alpha} h_\alpha \right) \\ &= \frac{\partial H}{\partial x^\alpha} \frac{\partial J}{\partial p_\alpha} - \frac{\partial H}{\partial p_\alpha} \frac{\partial J}{\partial x^\alpha} \\ &\equiv [H, J], \end{aligned} \tag{5.7}$$

where the final line serves to define the usual "Poisson Bracket". That's nice, because consider the flip side of the coin – the change in J as a particle moves along its trajectory is given by:

$$\begin{aligned} \frac{dJ}{dt} &= \frac{\partial J}{\partial x^\alpha} \dot{x}^\alpha + \frac{\partial J}{\partial p_\alpha} \dot{p}_\alpha \\ &= \frac{\partial J}{\partial x^\alpha} \frac{\partial H}{\partial p_\alpha} - \frac{\partial J}{\partial p_\alpha} \frac{\partial H}{\partial x^\alpha} \quad = -[H, J]. \end{aligned} \tag{5.8}$$

We have used the Hamiltonian equations of motion to rewrite $\dot{x}^\alpha$ and $\dot{p}_\alpha$ – so we are assuming that $x^\alpha(t)$ and $p_\alpha(t)$ take their dynamical form (i.e. that they satisfy the equations of motion). This gives us the time-derivative of J along the trajectory.

Now, the point: If we have a function J such that $[H, J] = 0$, then we know that:

- The Hamiltonian remains unchanged under the coordinate transformation implied by J: $\Delta H = [H, J] = 0$.
- The quantity J is a constant of the motion $\dot{J} = -[H, J] = 0$.

But keep in mind, J is intimately tied to a transformation! This is the Noetherian sentiment.

We have expanded our notion of transformation from its Lagrangian roots – now we can talk about transformations to coordinates and momenta as long as they can be generated (at least infinitesimally) by J.

5.2.1 Example

As a simple example to see how this works, let's take one dimensional motion under the potential $U(x)$ – the Hamiltonian is:

$$H = \frac{p^2}{2m} + U(x). \tag{5.9}$$

We will attempt to solve the PDE $[H, J] = 0$ for $J(x, p)$ a function of x and p. The vanishing Poisson Bracket is easy to write:

$$U'(x)\frac{\partial J}{\partial p} - \frac{p}{m}\frac{\partial J}{\partial x} = 0. \tag{5.10}$$

Now, how can we solve for J? Let's use separation of variables for starters (and finishers) – take an additive separation: $J = J_x(x) + J_p(p)$, then we have:

$$U'(x)J_p'(p) - \frac{p}{m}J_x'(x) = 0, \tag{5.11}$$

and to make the usual separation argument, we must divide by the product $(U'(x)p)$, and then we can simply set:

$$\frac{J_p'(p)}{p} = \alpha = \frac{J_x'(x)}{mU'(x)} \tag{5.12}$$

in order to solve (5.11). This gives $J_p'(p) = \frac{1}{2}\alpha p^2$ and $J_x(x) = \alpha\, m\, U(x)$ (plus constants of integration which just add overall constants to J – since J generates transformations via its derivatives, additive constants do not play an interesting role). Then we have:

$$J = \alpha m\left(\frac{p^2}{2m} + U(x)\right) = \alpha\, m\, H. \tag{5.13}$$

This tells us that any generator proportional to H is a constant of the motion (can you see what transformation is generated by this choice?) – no surprise, $[H, H] = 0$ automatically.

Suppose we instead consider multiplicative separation (familiar from, for example, E&M) – take $J(x,p) = J_x(x)\,J_p(p)$, then the Poisson Bracket relation reads:

$$J_x\,J_p'\,U' - \frac{p}{m}\left(J_p\,J_x'\right) = 0 \tag{5.14}$$

and dividing by $(J\,U'\,p)$ allows separation with constant α:

$$\frac{J_p'}{p\,J_p} = \alpha = \frac{J_x'}{m\,J_x\,U'} \tag{5.15}$$

and this has solution

$$J_x = \gamma\,e^{\alpha\,m\,U} \qquad J_p = \beta\,e^{\frac{\alpha\,p^2}{2}}, \tag{5.16}$$

for constants γ and β. Putting it together with an overall factor out front:

$$J = J_0\,e^{\alpha\,m\left(\frac{p^2}{2\,m}+U\right)} = J_0 e^{\alpha\,m\,H}. \tag{5.17}$$

We can now define the infinitesimal transformation for this J:

$$\begin{aligned} X &= x + \epsilon\,\frac{\partial J}{\partial p} = x + \underbrace{\epsilon\,J_0\,\alpha\,m}_{\equiv\beta}\,\frac{\partial H}{\partial p}\,e^{\alpha\,m\,H} = x + \beta\,\frac{p}{m}\,e^{\alpha\,m\,H} \\ P &= p - \epsilon\,\frac{\partial J}{\partial x} = p - \beta\,\frac{\partial H}{\partial x}\,e^{\alpha\,m\,H} = p - \beta\,U'\,e^{\alpha\,m\,H}, \end{aligned} \tag{5.18}$$

and, as advertised, if we rewrite the Hamiltonian in terms of these variables:

$$\begin{aligned} H &= \frac{1}{2\,m}\left(P + \beta\,U'\,e^{\alpha\,m\,H}\right)^2 + U(X - \beta\,\frac{p}{m}\,e^{\alpha\,m\,H}) \\ &= \frac{1}{2\,m}\left(P^2 + 2\,P\,\beta\,U'\,e^{\alpha\,m\,H}\right) + U(X) - U'(X)\,\beta\,\frac{p}{m}\,e^{\alpha\,m\,H} + O(\beta^2) \\ &= \frac{P^2}{2\,m} + U(X) + O(\beta^2), \end{aligned} \tag{5.19}$$

which was the point of the Poisson brackets in the first place.

As for the utility, once we have learned that, for example, $J = H$ has $\frac{dJ}{dt} = 0$ along the trajectory, then our work is highly simplified – let $J = E$, a number (which we recognize as the total energy) – then

$$E = \frac{p^2}{2\,m} + \frac{1}{2}\,k\,x^2, \tag{5.20}$$

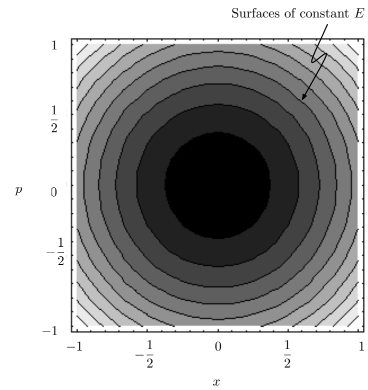

Figure 5.1: Constant energy contours for $\frac{1}{2}\,p^2 + \frac{1}{2}\,x^2 = E$.

and we can generate a geometrical relation between p and x, shown in Figure 5.1. It is no surprise here that isosurfaces of constant energy E form circles (for $m = k = 1$).

If we write H in terms of $\dot{x}$ by replacing p, we have simplified our problem – now we have a single first order ODE for $\dot{x}$ to solve. This is the functional utility of the constants of the motion we discover through Poisson Bracket PDE.

5.3 Hamiltonian for the Central Potential

Let's see how all of this plays out for our $U(r)$ potential – our goal is to find transformations that leave our Hamiltonian unchanged, and the associated constants of the motion. We have

$$H = \frac{1}{2}\,p_\alpha\,g^{\alpha\beta}\,p_\beta + U(r). \tag{5.21}$$

Now all we have to do is pick a J, a generator. Suppose we first ask: "What coordinate transformations can I have that set the new coordinates to a function purely of the old?". In other words, we want $X^\alpha = x^\alpha + \epsilon\,f^\alpha(x)$ where $f^\alpha(x)$ has no dependence on p. That tells us the form for J immediately – from $f^\alpha = \frac{\partial J}{\partial p_\alpha}$ we must have

$$J = p_\alpha\,f^\alpha(x) \tag{5.22}$$

which also tells us that the transformation for momenta follows $h_\alpha = -p_\beta \frac{\partial f^\beta}{\partial x^\alpha}$.

We have satisfied the "canonical infinitesimal transformation" requirement, now we want to deal with $\Delta H = 0$. I am concerned with finding specific constraints on f^α that will set $[H, J] = 0$, then I know that J is conserved. The Poisson bracket PDEs are

$$[H, J] = \frac{\partial H}{\partial x^\alpha} f^\alpha - \frac{\partial H}{\partial p_\alpha} p_\beta \frac{\partial f^\beta}{\partial x^\alpha} = 0 \tag{5.23}$$

and from the form of the Hamiltonian given above

$$\begin{aligned} \frac{\partial H}{\partial x^\alpha} &= \frac{1}{2} p_\beta \, p_\gamma \, g^{\gamma\beta}{}_{,\alpha} + U_{,\alpha} \\ \frac{\partial H}{\partial p_\alpha} &= p^\alpha. \end{aligned} \tag{5.24}$$

Then we have [1]

$$\begin{aligned} [H, J] &= \left(-\frac{1}{2} p^\gamma \, p^\beta \, g_{\gamma\beta,\alpha} + U_{,\alpha} \right) f^\alpha - g^{\gamma\delta} \, p_\delta \, p_\alpha \, f^\alpha_{\ ,\gamma} \\ &= -p^\alpha \, p^\beta \left(\frac{1}{2} g_{\alpha\beta,\gamma} \, f^\gamma + g_{\alpha\gamma} \, f^\gamma_{\ ,\beta} \right) + U_{,\alpha} \, f^\alpha = 0. \end{aligned} \tag{5.26}$$

Notice the two separate pieces to this: we must have *both* the term multiplying $p^\alpha \, p^\beta$ equal to zero, and $U_{,\alpha} \, f^\alpha = 0$ (which says that the coordinate transformation must be orthogonal to the force). The first term is a geometric statement, the second is physical.

In general relativity, where there is no potential, it is only the first term that counts, and vectors f^α satisfying this equation are called "Killing vectors". The side-constraint imposed by U can be dealt with after the general form implied by the PDE

$$\frac{1}{2} g_{\alpha\beta,\gamma} \, f^\gamma + g_{\alpha\gamma} \, f^\gamma_{\ ,\beta} = 0 \tag{5.27}$$

is satisfied.

The above equation, with some massaging is "Killing's equation", and more typically written as

$$f_{\mu;\nu} + f_{\nu;\mu} = 0 \qquad f_{\mu;\nu} \equiv f_{\mu,\nu} - \Gamma^\alpha_{\mu\nu} \, f_\alpha \tag{5.28}$$

a *tensor* statement – if true in one coordinate system, true in all.

[1] We use the result

$$g_{\mu\alpha} g^{\alpha\beta}{}_{,\gamma} g_{\beta\nu} = -g_{\mu\nu,\gamma}, \tag{5.25}$$

obtainable via the product rule for the ordinary derivative.

5.4 Example

Let's calculate a Killing vector in spherical coordinates. For the central potential, the extra portion of (5.26) reads: $\frac{\partial U}{\partial r} f^r = 0$ which we can accomplish simply by setting $f^r = 0$. For the other two components, I take:

$$f^\alpha \doteq \begin{pmatrix} 0 \\ f^\theta(r,\theta,\phi) \\ f^\phi(r,\theta,\phi) \end{pmatrix} \rightarrow f_\alpha \doteq \begin{pmatrix} 0 \\ r^2 f^\theta \\ r^2 \sin^2\theta \, f^\phi \end{pmatrix} \tag{5.29}$$

Then:

$$2 f_{2;2} = 2 \left(f_{2,2} - \Gamma^\sigma{}_{22} f_\sigma \right) \tag{5.30}$$

and

$$\Gamma^\sigma{}_{\alpha\beta} = \frac{1}{2} g^{\sigma\rho} \left(g_{\alpha\rho,\beta} + g_{\beta\rho,\alpha} - g_{\alpha\beta,\rho} \right) \tag{5.31}$$

so that $\Gamma^\sigma{}_{22} f_\sigma = \frac{1}{2} (f^2 g_{22,2}) = 0$, and we have

$$f_{2;2} = 2 f_{2,2} = 2 r^2 \frac{\partial f^\theta}{\partial \theta} \tag{5.32}$$

and so $f^\theta = f^\theta(\phi)$. By similar arguments, there is no r dependence in either f^θ or f^ϕ. We are left with:

$$\begin{aligned} f_{2;3} + f_{3;2} &= f_{2,3} + f_{3,2} - \Gamma_{\sigma 23} f^\sigma - \Gamma_{\sigma 32} f^\sigma \\ &= r^2 \frac{\partial f^\theta}{\partial \phi} + 2 r^2 \sin\theta \cos\theta \, f^\phi + r^2 \sin^2\theta \frac{\partial f^\phi}{\partial \theta} - 2 \left(\frac{1}{2} \left(g_{22,3} f^\theta + g_{33,2} f^\phi \right) \right) \\ &= r^2 \frac{\partial f^\theta}{\partial \phi} + r^2 \sin^2\theta \frac{\partial f^\phi}{\partial \theta}. \end{aligned} \tag{5.33}$$

Finally, we need

$$\begin{aligned} f_{3;3} = f_{3,3} - \Gamma_{\sigma 33} f^\sigma &= \frac{\partial f_3}{\partial \phi} - \left(-g_{33,2} f^2 \right) \\ &= r^2 \sin^2\theta \frac{\partial f^\phi}{\partial \phi} + \left(2 r^2 \sin\theta \cos\theta \, f^\theta \right). \end{aligned} \tag{5.34}$$

So our two (remaining) PDEs are

$$\begin{aligned} 0 &= r^2 \left(\frac{\partial f^\theta}{\partial \phi} + \sin^2\theta \frac{\partial f^\phi}{\partial \theta} \right) \\ 0 &= r^2 \sin\theta \left(\cos\theta \, f^\theta + \sin\theta \frac{\partial f^\phi}{\partial \phi} \right). \end{aligned} \tag{5.35}$$

The bottom equation can be integrated:

$$f^{\phi}(\theta,\phi) = -\cot\theta \int f^{\theta}\, d\phi + f^{\phi}(\theta), \tag{5.36}$$

and then the top is

$$\left(\int f^{\theta}\, d\phi + \frac{\partial f^{\theta}}{\partial \phi}\right) + \left(\sin^2\theta\, \frac{\partial f^{\phi}(\theta)}{\partial \theta}\right) = 0. \tag{5.37}$$

Because of the functional dependence here, we have a separation constant, each term must be equal to L or $-L$, but I'll set $L = 0$ and just solve, we find $f^{\phi} = F$, a constant, and $f^{\theta} = A\,\cos\phi + B\,\sin\phi$.

Our final form for the Killing vector f^{α} is

$$f^{\alpha} \doteq \begin{pmatrix} 0 \\ A\,\cos\phi + B\,\sin\phi \\ F + \cot\theta\,(B\,\cos\phi - A\,\sin\phi) \end{pmatrix} \tag{5.38}$$

about which we shall say more next time.