

Lagrangian for Central Potentials

Lecture 2

Physics 411
Classical Mechanics II

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Here we will review the Lagrange formulation in preparation for the study of the central potential problem. Our ultimate goal is to shift from the specific sorts of notations used in introductory cases (for example, spherical coordinates), to a more abstract notation appropriate to the study of general relativity. This also serves as a short review of both Lagrangians in general and Keplerian orbits in particular in the classical setting – if we are to understand the changes to the motion of particles on the GR side, it behooves us to recall the motion in the “normal” case.

Along the notation-shifting lines, we will introduce some basic tensor operations, but we will see a lot more of this later on. We just need to become comfortable with summation notation for now.

2.1 Lagrangian and Equations of Motion

A Lagrangian is the integrand of an action. In classical mechanics, the Lagrangian leading to Newton’s second law reads, in Cartesian coordinates ¹:

$$L = \frac{1}{2} m \underbrace{(\mathbf{v}(t) \cdot \mathbf{v}(t))}_{\equiv T} - U(\mathbf{x}(t)), \quad (2.1)$$

where we view x , y and z as functions of a parameter t which we normally interpret as “time”. The first term is the kinetic energy, the second is the potential energy. The ultimate goal of classical mechanics is to find the

¹I will refer to the “vector” (more appropriately, the coordinate differential is the vector) of coordinates as $\mathbf{x} \equiv x\hat{x} + y\hat{y} + z\hat{z}$, and its time-derivative (velocity) as: $\mathbf{v} = \frac{d\mathbf{x}}{dt}$.

trajectory of a particle under the influence of a force. Physically, we control the description of the system by specifying the particle mass, form of the force, and boundary conditions (particle starts from rest, particle moves from point a to point b , etc.). Mathematically, we use the equations of motion derived from the Lagrangian, together with the boundary conditions to determine the curve $\mathbf{x}(t) \doteq (x(t), y(t), z(t))$ through three-dimensional space.

The Euler-Lagrange equations, come from an extremization in the variational calculus sense, of the action:

$$S[x(t)] = \int L(\mathbf{x}(t), \mathbf{v}(t)) dt, \quad (2.2)$$

and provide the ODE structure of interest, a set of three second order differential equations:

$$\boxed{\frac{d}{dt} \frac{\partial L}{\partial \mathbf{v}} - \frac{\partial L}{\partial \mathbf{x}} = 0 \longrightarrow m \ddot{\mathbf{x}}(t) = -\nabla U(\mathbf{x}).} \quad (2.3)$$

The advantage of the action approach, and the Lagrangian in particular, is that the equations of motion can be obtained for any coordinate representation of the kinetic energy and potential. Although it is easy to define and verify the correctness of the Euler-Lagrange equations in Cartesian coordinates, they are not necessary to the formulation of valid equations of motion for systems in which Cartesian coordinates are less physically and mathematically useful.

For example, if we write the Lagrangian in cylindrical coordinates:

$$L(s, \phi, z) = L(\mathbf{x}(s, \phi, z)) = \frac{1}{2} m (\dot{s}^2 + s^2 \dot{\phi}^2 + \dot{z}^2) - U(s, \phi, z) \quad (2.4)$$

then we again get three equations of motion, expressed as

$$\begin{aligned} 0 &= \frac{d}{dt} \frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} \\ 0 &= \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} \\ 0 &= \frac{d}{dt} \frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z}. \end{aligned} \quad (2.5)$$

2.1.1 Examples

In one dimension, we can consider the Lagrangian $L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k (x - a)^2$, appropriate to a spring potential with spring constant k and equilibrium

spacing a . Then the Euler-Lagrange equations give:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \longrightarrow m \ddot{x} + k(x - a) = 0. \quad (2.6)$$

Notice that for a real physical problem, the above equation of motion is not enough – we also need to specify two initial conditions. We can phrase this choice in terms of boundaries in time at $t = t_0$ and $t = t_f$ (particle starts at 1 m from the origin at $t = 0$ and ends at 2 m from the origin at $t = 10$ s), or as an initial position and velocity (particle starts at equilibrium position with speed 5 m/s) – there are other choices as well, depending on our particular experimental setup.

In two dimensions, we could express a radial spring potential:

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k (\sqrt{x^2 + y^2} - a)^2, \quad (2.7)$$

giving us two Euler-Lagrange equations:

$$\begin{aligned} 0 &= m \ddot{x} + \frac{k x (\sqrt{x^2 + y^2} - a)}{\sqrt{x^2 + y^2}} \\ 0 &= m \ddot{y} + \frac{k y (\sqrt{x^2 + y^2} - a)}{\sqrt{x^2 + y^2}}. \end{aligned} \quad (2.8)$$

Suppose we want to transform to two-dimensional polar coordinates via $x = s(t) \cos \phi(t)$ and $y = s(t) \sin \phi(t)$ – we can write the above in terms of the derivatives of $s(t)$ and $\phi(t)$ and solve to get:

$$\begin{aligned} \ddot{s} &= -\frac{k}{m} (s - a) + s \dot{\phi}^2 \\ \ddot{\phi} &= -\frac{2 \dot{\phi} \dot{s}}{s}. \end{aligned} \quad (2.9)$$

Or, we could take advantage of the coordinate-independence of the Lagrangian to rewrite L directly in terms of $s(t)$ and $\phi(t)$, where it becomes

$$L = \frac{1}{2} m (\dot{s}^2 + s^2 \dot{\phi}^2) - \frac{1}{2} k (s - a)^2. \quad (2.10)$$

Then the Euler-Lagrange analogues of (2.5) become:

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} &= m \ddot{s} - m s \dot{\phi}^2 + k (s - a) = 0 \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} &= m (2 s \dot{s} \dot{\phi} + s^2 \ddot{\phi}) = 0, \end{aligned} \quad (2.11)$$

precisely the above, and well worth the effort of rewriting just the Lagrangian.

The above also provides insight into the motivation for using different coordinate systems – notice that in the cylindrical case, the ϕ coordinate does not appear in the Lagrangian at all, only $\dot{\phi}$ shows up. Then we know from the equations of motion that:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = 0 \longrightarrow \frac{\partial L}{\partial \dot{\phi}} = \text{constant of the motion.} \quad (2.12)$$

When possible, this type of observation can be useful in actually solving the equations of motion. Finding (or constructing) a coordinate system in which one or more of the coordinates do not appear is one of the goals of Hamilton-Jacobi theory.

2.2 Lagrangian for a Central Potential

We want to find the parametrization of a curve $\mathbf{x}(t)$ corresponding to motion under the influence of a central potential. Central potentials depend only on a particle's distance from some origin, so they take the specific form: $U(x, y, z) = U(r)$ with $r^2 \equiv x^2 + y^2 + z^2$. We know, then, that the associated force will be directed either towards or away from the particle (since according to Newton's second law, $\nabla U \sim \hat{\mathbf{x}}$, the usual result from electrostatics, for example).

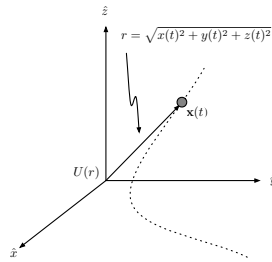


Figure 2.1: A particle trajectory, the goal is to find the curve $\mathbf{x}(t)$, which, using the equations of motion, we interpret as the trajectory of a particle.

The Lagrangian for the problem is, as always

$$L = T - U = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(r). \quad (2.13)$$

The move to spherical coordinates is obvious, so define (r, θ, ϕ) in terms of (x, y, z) :

$$\begin{aligned}x &= r \sin \theta \cos \phi \\y &= r \sin \theta \sin \phi \\z &= r \cos \theta.\end{aligned}\tag{2.14}$$

Then the kinetic term in the Lagrangian will be made up of the following derivatives:

$$\begin{aligned}\dot{x}^2 &= \left(\dot{r} \sin \theta \cos \phi + r \dot{\theta} \cos \theta \cos \phi - r \dot{\phi} \sin \theta \sin \phi \right)^2 \\ \dot{y}^2 &= \left(\dot{r} \sin \theta \sin \phi + r \dot{\theta} \cos \theta \sin \phi + r \dot{\phi} \sin \theta \cos \phi \right)^2 \\ \dot{z}^2 &= \left(\dot{r} \cos \theta - r \dot{\theta} \sin \theta \right)^2,\end{aligned}\tag{2.15}$$

and without changing anything, we have

$$\boxed{L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) - U(r).}\tag{2.16}$$

The equations of motion for this Lagrangian are the usual:

$$\begin{aligned}0 &= \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = m \ddot{r} - m r \left(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right) + \frac{\partial U}{\partial r} \\ 0 &= \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = m r^2 \ddot{\theta} + 2 m r \dot{r} \dot{\theta} - m r^2 \sin \theta \cos \theta \dot{\phi}^2 \\ 0 &= \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = m r^2 \sin^2 \theta \ddot{\phi} + 2 m r \sin^2 \theta \dot{r} \dot{\phi} + 2 m r^2 \sin \theta \cos \theta \dot{\theta} \dot{\phi}.\end{aligned}\tag{2.17}$$

Now none of us would have gone this far without using some simplifying assumptions, but the form of the above is interesting. Notice that there is only one term that we would associate with the physical environment (only one term involves the potential), the rest are somehow residuals of the coordinate system we are using.

More interesting than that is the structure of the equations of motion – everything that isn't \ddot{X} looks like $f(r, \theta) \dot{X} \dot{Y}$ (here, $X, Y \in (r, \theta, \phi)$). That is somewhat telling, and says more about the structure of the Lagrangian than anything else. Setting aside the details of spherical coordinates and central potentials, we can gain insight into the classical Lagrangian by looking at it from a slightly different point of view – one that will allow us to generalize it appropriately to both special relativity and general relativity. We will return to the central potential after this short aside.

2.3 The Metric

And so, innocuously, begins our journey. My first goal here is to use a little matrix notation. Refer back to (2.13), I am free to write the kinetic term as:

$$L = \frac{1}{2} m \begin{pmatrix} \dot{x} & \dot{y} & \dot{z} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} - U(r) \quad (2.18)$$

just as I can write

$$L = \frac{1}{2} m \begin{pmatrix} \dot{r} & \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \begin{pmatrix} \dot{r} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} - U(r). \quad (2.19)$$

You can't stop me.

But while it takes up a lot more space on the board, this is not so trivial a statement. Think of it this way – consider two points infinitesimally close together in the two coordinate systems,

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 && \text{Pythagoras} \\ ds^2 &= dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) && \text{also Pythagoras} \end{aligned} \quad (2.20)$$

this is just the Pythagorean theorem in two different coordinate systems. The *distance* between the two points is the same in both, that can't change, but the representation is different.

Well, again, let me be pedantic, we can write these as

$$ds^2 = \begin{pmatrix} dx & dy & dz \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \quad (2.21)$$

and

$$ds^2 = \begin{pmatrix} dr & d\theta & d\phi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix}. \quad (2.22)$$

Huh! That looks awfully familiar, and this isn't really so surprising. After all, velocity is intimately related to infinitesimal displacements.

The notation point I want to make is this: “We call the matrix in the above ‘the metric’”. It is often represented not as a matrix but as a “second-rank tensor” and denoted $g_{\mu\nu}$. It tells us, given a coordinate system, how to measure distances. In classical mechanics, we usually go the other way around, as we have done here – we figure out how to measure distances in the new coordinates and use that to find $g_{\mu\nu}$.

Let me make one more shift. If I call the vectors appearing in the above equation dx^μ , then I can define the “Einstein Summation Notation”

$$ds^2 = \begin{pmatrix} dx & dy & dz \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \equiv dx^\mu g_{\mu\nu} dx^\nu \quad (2.23)$$

and this works like you would expect from the above. We take each repeated index and sum it over the dimension of the space. Rename $x = x^1$, $y = x^2$, $z = x^3$, then

$$dx^\mu g_{\mu\nu} dx^\nu = dx^1 g_{11} dx^1 + dx^2 g_{22} dx^2 + dx^3 g_{33} dx^3 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \quad (2.24)$$

In general, there would be more terms in the sum, the diagonal form of $g_{\mu\nu}$ simplified life (here). The same equation holds if we take $g_{\mu\nu}$ to be the matrix defined in (2.22) and $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$.

In Einstein summation notation, we sum over repeated indices where one is up, one is down (objects like $g_{\mu\nu} dx_\mu$ are nonsense and will never appear). The repeated index, because it takes on all values $1 \rightarrow D$ (in this case, $D = 3$ dimensions) has no role in labelling a component, and so can be renamed as we wish, leading to statements like:

$$ds^2 = dx^\mu g_{\mu\nu} dx^\nu = dx^\gamma g_{\gamma\nu} dx^\nu = dx^\alpha g_{\alpha\beta} dx^\beta. \quad (2.25)$$

And finally, general properties of the metric: 1. The metric is symmetric, this is a convenient notational device here, we have no reason to expect $dx dy \neq dy dx$ in a line element. 2. It does not have to be diagonal. 3. It can depend (as with the spherical metric) on position.

2.4 Lagrangian Redux

Returning to the central potential Lagrangian, we can write:

$$L = \frac{1}{2} m \dot{x}^\mu g_{\mu\nu} \dot{x}^\nu - U(r). \quad (2.26)$$

That's sensible, *and* I don't even have to tell you which coordinates I mean. We can vary as before, keeping in mind that it is possible (as in the spherical case) that $g_{\mu\nu}(x^\mu)$, i.e. the metric depends on the coordinates.

The equations of motion, written in tensor form, are

$$0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^\alpha} - \frac{\partial L}{\partial x^\alpha}, \quad (2.27)$$

and let's just review the notation. We need to take a \dot{x}^α derivative of the kinetic term:

$$\frac{\partial}{\partial \dot{x}^\alpha} \left(\frac{1}{2} m \dot{x}^\mu g_{\mu\nu} \dot{x}^\nu \right) = \frac{1}{2} m \left(\frac{\partial \dot{x}^\mu}{\partial \dot{x}^\alpha} g_{\mu\nu} \dot{x}^\nu + \dot{x}^\mu g_{\mu\nu} \frac{\partial \dot{x}^\nu}{\partial \dot{x}^\alpha} \right), \quad (2.28)$$

but it is evident that

$$\frac{\partial \dot{x}^\mu}{\partial \dot{x}^\alpha} = \delta_\alpha^\mu \quad (2.29)$$

so that:

$$\begin{aligned} \frac{1}{2} m \left(\frac{\partial \dot{x}^\mu}{\partial \dot{x}^\alpha} g_{\mu\nu} \dot{x}^\nu + \dot{x}^\mu g_{\mu\nu} \frac{\partial \dot{x}^\nu}{\partial \dot{x}^\alpha} \right) &= \frac{1}{2} m (\delta_\alpha^\mu g_{\mu\nu} \dot{x}^\nu + \dot{x}^\mu g_{\mu\nu} \delta_\alpha^\nu) \\ &= \frac{1}{2} m (g_{\alpha\nu} \dot{x}^\nu + \dot{x}^\mu g_{\mu\alpha}) \end{aligned} \quad (2.30)$$

(using $g_{\mu\nu} \delta_\alpha^\nu = g_{\mu\alpha}$). Because in the first term, ν is a "dummy index" (it is summed over) and in the second, μ is a dummy, we can relabel these however we like. In addition, we can use the symmetry property of $g_{\mu\nu}$ to write, finally:

$$\frac{1}{2} m (g_{\alpha\nu} \dot{x}^\nu + \dot{x}^\nu g_{\alpha\nu}) = m g_{\alpha\nu} \dot{x}^\nu. \quad (2.31)$$

That's just the flavor of the sorts of calculation we will be doing over (and over) again.

The equations of motion now read:

$$\begin{aligned} \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^\alpha} - \frac{\partial L}{\partial x^\alpha} \right) &= m \frac{d}{dt} (g_{\alpha\nu} \dot{x}^\nu) - \frac{1}{2} m \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \dot{x}^\mu \dot{x}^\nu + \frac{\partial U}{\partial x^\alpha} \\ &= m \frac{\partial g_{\alpha\nu}}{\partial x^\gamma} \dot{x}^\gamma \dot{x}^\nu + m g_{\alpha\nu} \ddot{x}^\nu - \frac{1}{2} m \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \dot{x}^\mu \dot{x}^\nu + \frac{\partial U}{\partial x^\alpha} \\ &= m g_{\alpha\nu} \ddot{x}^\nu + m \dot{x}^\nu \dot{x}^\gamma \left(\frac{\partial g_{\alpha\nu}}{\partial x^\gamma} - \frac{1}{2} \frac{\partial g_{\gamma\nu}}{\partial x^\alpha} \right) + \frac{\partial U}{\partial x^\alpha} \end{aligned} \quad (2.32)$$

and now for a little sleight-of-hand which you will prove in your homework – notice in this last line that the second term has $\dot{x}^\nu \dot{x}^\gamma$, which is symmetric in $\nu \leftrightarrow \gamma$. Using your result from homework, we have:

$$m \dot{x}^\nu \dot{x}^\gamma \left(\frac{\partial g_{\alpha\nu}}{\partial x^\gamma} - \frac{1}{2} \frac{\partial g_{\gamma\nu}}{\partial x^\alpha} \right) = m \dot{x}^\nu \dot{x}^\gamma \left(\frac{1}{2} \frac{\partial g_{\alpha\nu}}{\partial x^\gamma} + \frac{1}{2} \frac{\partial g_{\alpha\gamma}}{\partial x^\nu} - \frac{1}{2} \frac{\partial g_{\gamma\nu}}{\partial x^\alpha} \right) \quad (2.33)$$

and we can write the equations of motions as

$$m g_{\alpha\nu} \ddot{x}^\nu + m \dot{x}^\nu \dot{x}^\gamma \left(\frac{1}{2} \frac{\partial g_{\alpha\nu}}{\partial x^\gamma} + \frac{1}{2} \frac{\partial g_{\alpha\gamma}}{\partial x^\nu} - \frac{1}{2} \frac{\partial g_{\gamma\nu}}{\partial x^\alpha} \right) = - \frac{\partial U}{\partial x^\alpha}. \quad (2.34)$$

The term in parenthesis appears a lot, and is given a special name – it is called the “connection coefficient for the metric $g_{\mu\nu}$ ” and is denoted $\Gamma_{\alpha\gamma\nu}$.

We have, in its final form,

$$\boxed{m g_{\alpha\nu} \ddot{x}^\nu + m \Gamma_{\alpha\nu\gamma} \dot{x}^\nu \dot{x}^\gamma = - \frac{\partial U}{\partial x^\alpha}.} \quad (2.35)$$

The terms in the above that are not explicitly \ddot{x} -or- U related are quadratic in the first derivatives, just as we saw explicitly in spherical coordinates.