Stress Tensors, Particles and Fields

Lecture 19

Physics 411
Classical Mechanics II

October 10th, 2007

We continue looking at the energy-momentum tensor (the four-dimensional extension of stress tensors), first establishing the usual interpretations for a single free particle, and then making the connection to the field tensors we began to develop last time. Along the way, we introduce the Lagrangian density for point particles, and this will prove useful when we actually attempt to couple theories to matter.

Then, the first-order form for the action is developed, we see that in this form, actions are relatively easy to guess, and we will eventually use this to develop both the standard vector field theory (E&M) and with some modification, general relativity itself.

19.1 Energy-Momentum Tensor for Particles

Because it is the fundamental object of interest for matter coupling, and also because it sheds some light on the field energy-momentum tensor, we want to connect the $T^{\mu\nu}$ that comes from particles themselves to the field version. Consider the usual action for free particles in special relativity:

$$\boxed{S = -m\,c \int \sqrt{-\dot{x}^\mu \, g_{\mu\nu} \, \dot{x}^\nu} \, ds} \tag{19.1}$$

where dots refer to differentiation by s. If we take s to be the proper time, then this reduces to $S = -m\,c^2 \int ds$, and we can expand ds in the low-velocity limit to recover our three-dimensional $L = \frac{1}{2}\,m\,v^2$. Since we are talking about free particles in affine parametrization (proper time), the quantity $\sqrt{-\dot{x}^\mu \, g_{\mu\nu} \, \dot{x}^\nu}$ is actually constant (with value c for Minkowski).

The problem is, we do not have a four-volume integral, so our notion of the energy-momentum tensor is difficult to define. We do know that the particle is moving along some trajectory, $x^\mu(s)$, and then we can view $\rho \longrightarrow m\,\delta^4(x^\mu - x^\mu(s))$ – the four-dimensional delta function is itself a density, so there is a scalar here:

$$S_p \equiv -m\,c \int\int ds d\tau\, \sqrt{-\dot{x}^\mu\, g_{\mu\nu}\, \dot{x}^\nu}\, \delta^4(x^\mu - x^\mu(s)). \tag{19.2}$$

The energy-momentum tensor *density* $\mathcal{T}^{\mu\nu} = \sqrt{-g}\, T^{\mu\nu}$ can be obtained from our general form

$$\frac{1}{2}\sqrt{-g}\, T^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} \longrightarrow \mathcal{T}^{\mu\nu} = -2\,\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} \tag{19.3}$$

with $\mathcal{L}$ the Lagrange density, given for the particle by

$$\mathcal{L} = -\int ds\, m\, c\, \sqrt{-\dot{x}^\mu\, g_{\mu\nu}\, \dot{x}^\nu}\, \delta^4(x^\mu - x^\mu(s)) \tag{19.4}$$

so that the tensor density is

$$\begin{aligned} \mathcal{T}^{\mu\nu} &= \int ds\, \frac{m\, c\, \dot{x}^\mu\, \dot{x}^\nu}{\sqrt{-\dot{x}^\mu\, g_{\mu\nu}\, \dot{x}^\nu}} \delta(t - t(s))\, \delta^3(\mathbf{x} - \mathbf{x}(t)) \\ &= m\, \dot{x}^\mu\, \dot{x}^\nu\, \frac{ds}{dt}\, \delta^3(\mathbf{x} - \mathbf{x}(s(t))), \end{aligned} \tag{19.5}$$

where we have transformed the integral over s to one over t, used the $\delta(t - t(s))$ to perform the integration, and then transformed back to the s parametrization. We have also used the invariance of the interval (in proper time parametrization) $\sqrt{-\dot{x}^\mu\, g_{\mu\nu}\, \dot{x}^\nu} = c$. The three-dimensional delta function just says that we take the stress tensor for the particle and evaluate it along the particle trajectory in three dimensions, with appropriate Lorentz factor. Note that for Minkowksi space, $\frac{ds}{dt}$ is given by

$$c^2\, ds^2 = c^2\, dt^2 - dx^2 - dy^2 - dz^2 \longrightarrow \frac{ds}{dt} = \sqrt{1 - \frac{v^2}{c^2}} \tag{19.6}$$

where v is the time-parametrized particle velocity.

If we are in the rest-frame of the particle, $\frac{ds}{dt} = 1$, and $\frac{dx^0}{ds} = \frac{c\, dt}{ds} = c$, so the energy momentum tensor density has only one term, $\mathcal{T}^{00} = m\, c^2$ (along its trajectory), which is the energy density of the particle. Take the temporal-spatial three-vector (not necessarily in the rest-frame), we have

$$\mathcal{T}^{0j} = m\, c\, \frac{dt}{ds}\, \frac{dx^j}{ds}\, \frac{ds}{dt} \doteq \frac{m\, c\, \mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{19.7}$$

which is, modulo the factor of c in the numerator, the three-velocity portion of the four-velocity (with v written in "usual" $\frac{d}{dt}$ form). Indeed, the full four-vector is,

$$\boxed{\mathcal{T}^{0\mu} \doteq \begin{pmatrix} \frac{m\,c^2}{\sqrt{1-\frac{v^2}{c^2}}} \\ \frac{m\,c\,\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}} \end{pmatrix} = c\,p^\mu.} \tag{19.8}$$

Precisely the energy-momentum tensor of special relativity. I have omitted the spatial delta function that ensures we are evaluating the particle along its trajectory, this seems obvious for a point particle.

Finally, the spatial-spatial components represent a sort of generalized (directional) kinetic energy, as they should if they are to be interpreted as momentum flux density.

19.2 Energy and Momenta for Fields

If we view the $T^{00} = \mathcal{E}$ entry as energy density for the fields (as we saw for the scalar field last time), then the integrals associated with $T^{\mu\nu}{}_{;\nu} = 0$ give a natural interpretation to the other elements:

$$\frac{\partial}{\partial t}\int \mathcal{E}\,d\tau = -\oint c\,T^{0j}\,da_j \tag{19.9}$$

represents the time-derivative (on the left) of the energy enclosed in a volume, the total change of energy in the volume in time. On the right is energy passing out through a surface, (i.e. $c\,T^{0j}$ is energy per unit area per unit time) – the energy flux density. If we compare with the particle case, we see, from (19.8) that $\frac{1}{c}\,\mathcal{T}^{0j} \sim \mathfrak{p}$ is the momentum density, so that it is reasonable to associate the (physical) field spatial momenta with

$$P^j = \frac{1}{c}\int T^{0j}\,d\tau, \tag{19.10}$$

as we did with the scalar field.

The purely spatial components give another continuity statement

$$\underbrace{\frac{\partial}{\partial t}\int \frac{1}{v}\,T^{0j}\,d\tau}_{=\frac{\partial}{\partial t}P^j} = -\oint T^{jk}\,da_k, \tag{19.11}$$

so that the spatial components of $T^{0\mu}$ also change with time, with flux density given by T^{jk} which has units of force per unit area. Thinking back to E&M, we associate the T^{00} component with energy density and the T^{0j} component with the Poynting vector (or more generally, momentum density for the fields). Similarly, the change in momentum density comes from the Maxwell stress tensor, which forms a momentum flux density. We will see all of this explicitly later on, but the point is – the energy and momentum conservation statements for a field theory are entirely encapsulated in the vanishing of the divergence of the energy-momentum tensor $T^{\mu\nu}$.

19.3 First Order Action

We now switch gears a bit, and return to the treatment of actions, this time by way of a "Hamiltonian". I use this term loosely here, what we will really be doing is a series of Legendre transforms. Hamiltonians, in field theory, still represent a specific breakup of time and space, so the procedure below is more of a Hamiltonian-ization of the action, and not what most people would call the Hamiltonian of a field theory.

The Lagrange density is the most natural way to define the action for a field theory, coming to us, as it does, directly from a discrete multi-particle Lagrangian. But $\bar{\mathcal{L}}$ does not necessarily provide the most useful description of the fields, it leads to PDE's that are second order in the "coordinates" (time and space, now).

In our usual classical mechanics setting, it is possible to turn a single second order equation of motion into two first-order equations through the Hamiltonian. The same can be done on the field side. Consider the free scalar field action

$$S = \int d\tau \, \sqrt{-g} \, \underbrace{\left(\frac{1}{2} \, \phi_{,\mu} \, g^{\mu\nu} \, \phi_{,\nu} \right)}_{=\bar{\mathcal{L}}}, \tag{19.12}$$

and notice the correspondence between this and a free particle action $S = \int ds \, (\frac{1}{2} \, \dot{x}^\mu \, g_{\mu\nu} \, \dot{x}^\nu)$ – what if we imagine the replacement $\dot{x}^\mu \to \phi_{,\mu}$? That would make the "conjugate momenta" for the field

$$\pi^\alpha \equiv \frac{\partial \bar{\mathcal{L}}}{\partial \phi_{,\alpha}} \tag{19.13}$$

and we could get a Hamiltonian just by taking the Legendre transform

$$\bar{\mathcal{H}} = \phi_{,\alpha}\, \pi^\alpha - \bar{\mathcal{L}}. \tag{19.14}$$

Then, just as in classical mechanics, we can form the so-called "first-order action" (because it leads to first-order field equation) by inverting the Legendre transform

$$\boxed{S = \int d\tau \,\sqrt{-g}\, \left(\phi_{,\alpha}\, \pi^\alpha - \bar{\mathcal{H}}\right).} \tag{19.15}$$

Aside from a slight change of notation, there is a relatively large shift in focus here – a Hamiltonian is expressed entirely in terms of the momenta and coordinates (field momenta π and field ϕ here), and the action written in the above form has an explicit term depending on $\phi_{,\alpha}$ but no other hidden $\phi_{,\alpha}$ dependence.

For our scalar action, we have

$$\frac{\partial \bar{\mathcal{L}}}{\partial \phi_{,\alpha}} = g^{\alpha\nu}\, \phi_{,\nu} \qquad \bar{\mathcal{H}} = \frac{1}{2}\, \pi^\alpha\, g_{\alpha\beta}\, \pi^\beta \tag{19.16}$$

so that the first-order action is

$$S[\phi, \pi] = \int d\tau \,\sqrt{-g}\, \left(\phi_{,\alpha}\, \pi^\alpha - \frac{1}{2}\, \pi^\alpha\, g_{\alpha\beta}\, \pi^\beta\right). \tag{19.17}$$

Now we can vary w.r.t. ϕ and π independently – the ϕ variation gives the usual:

$$\begin{aligned} \frac{\delta S}{\delta \phi} &= \int d\tau \,\sqrt{-g}\, \pi^\alpha\, \delta\phi_{,\alpha} \\ &= \underbrace{\int d\tau \,\left(\sqrt{-g}\, \pi^\alpha\, \delta\phi\right)_{;\alpha}}_{=0} - \int d\tau \,\left(\sqrt{-g}\, \pi^\alpha\right)_{;\alpha}\, \delta\phi, \end{aligned} \tag{19.18}$$

and the second term tells us that $\pi^\alpha{}_{;\alpha} = 0$ (remember that $(\sqrt{-g})_{;\alpha} = 0$).

Varying the action w.r.t. π^α gives

$$\frac{\delta S}{\delta \pi^\alpha} = \int d\tau \,\sqrt{-g}\, \left(\phi_{,\alpha} - g_{\alpha\beta}\, \pi^\beta\right)\, \delta\pi^\alpha \tag{19.19}$$

and for this to be zero for arbitrary $\delta\pi^\alpha$, we must have:

$$\phi_{,\alpha} - g_{\alpha\beta}\, \pi^\beta = 0 \longrightarrow \pi^\beta = g^{\alpha\beta}\, \phi_{,\alpha} \tag{19.20}$$

which reproduces the definition of π^β. Putting these two together, our two first-order field equations are

$$\left.\begin{array}{l} \pi^\beta = g^{\alpha\beta}\,\phi_{,\alpha} \\ \pi^\alpha{}_{;\alpha} = 0 \end{array}\right\} \longrightarrow \phi_{;\ \alpha}^{\alpha} = 0. \tag{19.21}$$

So what? Well, the advantage here is that the action for a free scalar field is, at this point, easy to guess. Suppose we knew from the start that a free field has $\bar{\mathcal{H}} = \frac{1}{2}\,\pi^\alpha\,\pi_\alpha$ which is a pretty reasonable form. Then we are told that, via Legendre, we must introduce derivatives of a scalar field (those are first rank tensors), the resulting action is then fixed. It's clear from $\pi^\alpha\,\pi_\alpha$ that we are dealing with scalar fields, and then a term like $\phi_{,\alpha}\,\pi^\alpha$ is obvious.