Newtonian Gravity and Special Relativity

Lecture 12

Physics 411
Classical Mechanics II

Monday, September 24th, 2007

It is interesting to note that under Lorentz transformation, while electric and magnetic fields get mixed together, the force on a particle is identical in magnitude and direction in the two frames related by the transformation. Indeed, that was the motivation for looking at the manifestly relativistic structure of Maxwell's equations. The idea was that Maxwell's equations and the Lorentz force law are automatically in accord with the notion that observations made in inertial frames are physically equivalent, even though observers may disagree on the names of these forces (electric or magnetic).

Today, we will look at a force (Newtonian gravity) that does not have the property that different inertial frames agree on the physics. That will lead us to an obvious correction that is, qualitatively, a prediction of (linearized) general relativity.

12.1 Newtonian Gravity

We start with the experimental observation that for a particle of mass M and another of mass m, the force of gravitational attraction between them, according to Newton, is (see Figure 12.1):

$$\mathbf{F} = -\frac{G\,M\,m}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}} \qquad \boldsymbol{\mathscr{r}} \equiv \mathbf{r} - \mathbf{r}\,'. \tag{12.1}$$

From the force, we can, by analogy with electrostatics, construct the Newtonian gravitational field and its associated point potential:

$$\mathbf{G} = -\frac{G\,M}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}} = -\nabla\underbrace{\left(-\frac{G\,M}{\mathscr{r}}\right)}_{\equiv\phi}. \tag{12.2}$$

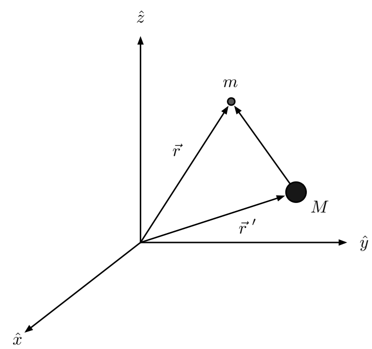

Figure 12.1: Two particles interacting via the Newtonian gravitational force.

By applying the Laplacian (or using our eyes), we see that the the potential field ϕ satisfies

$$\nabla^2 \phi = 4\,\pi\,G\,\rho_m(\mathbf{r}), \tag{12.3}$$

where $\rho_m(\mathbf{r})$ is the density of a distribution of *mass*.

Comparing this to the electrostatic potential:

$$\nabla^2 V = -\frac{\rho_e(\mathbf{r})}{\epsilon_0}, \tag{12.4}$$

we can map electrostatic results to gravitational results by the replacement:

$$\boxed{\rho_e \longrightarrow -\epsilon_0\,\rho_m\,(4\pi G).} \tag{12.5}$$

Notice that we already sense a clash with special relativity – the Newtonian gravitational potential, like the pure electrostatic one, responds instantaneously to changes in the source distribution – that is not the sort of causal behavior we expect in a true relativistic theory.

12.2 Lines of Mass

Consider an infinite rod with uniform mass density λ_0 (mass per unit length, here). What is the gravitational field associated with this configuration? If this were an electrostatics problem, the electric field would be:

$$\mathbf{E} = \frac{\lambda_0}{2\,\pi\,\epsilon_0\,s}\,\hat{\mathbf{s}} \tag{12.6}$$

and we apply our map (12.5) to obtain the relevant gravitational field:

$$\mathbf{G} = -\frac{2\,G\,\lambda_0}{s}\,\hat{\mathbf{s}}. \tag{12.7}$$

The force on a particle of mass m a distance s away from the line is, then, $\mathbf{F} = -\frac{2\,m\,G\,\lambda_0}{s}\,\hat{\mathbf{s}}$. Referring to Figure 12.2, we can write the force in terms of the local Cartesian axes for a particle in the plane of the paper as shown.

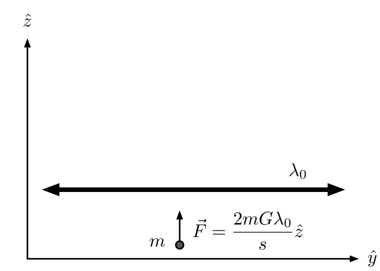

Figure 12.2: A particle of mass m feels a force in the $\hat{z}$ direction due to the infinite line of uniform mass.

12.2.1 Moving Line of Mass

If we take the line of uniform mass from above and pull it to the right with speed $\bar{v}$ as measured in a frame $\bar{\mathcal{O}}$ (the rest frame of the laboratory), then the only change to the force on a test particle in the lab is in the Lorentz contraction of the mass per unit length of the source: $\bar{\lambda} = \gamma_{\bar{v}}\,\lambda_0$. Newtonian gravity has nothing to say about sourcing provided by the relative motion of masses, the only source for the theory is the mass density itself. In our lab frame $\bar{\mathcal{O}}$, then, we have the observed force on a test mass moving at speed $\bar{u}$ to the right:

$$\bar{\mathbf{F}} = \frac{2\,m\,G\,\gamma_{\bar{v}}\,\lambda_0}{s}\,\hat{\mathbf{z}} \qquad \gamma_{\bar{v}} \equiv \frac{1}{\sqrt{1-\frac{\bar{v}^2}{c^2}}}. \tag{12.8}$$

The situation is shown in Figure 12.3.

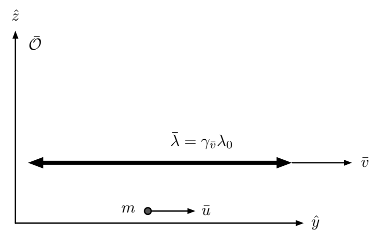

Figure 12.3: A uniform line mass of rest length λ_0 moves at constant speed $\bar{v}$ to the right in a lab.

12.2.2 Analysis in the Rest Frame

In the rest frame of the test mass m (which we will call $\mathcal{O}$), the moving line mass that has speed $\bar{v}$ relative to $\bar{\mathcal{O}}$ has speed:

$$v = \frac{\bar{v} - \bar{u}}{1 - \frac{\bar{u}\,\bar{v}}{c^2}} \tag{12.9}$$

relative to $\mathcal{O}$ (since in this frame, the lab is moving to the left with speed $\bar{u}$). We can then write the force on the stationary test mass (in this frame) as

$$\mathbf{F} = \frac{2\,m\,G\,\gamma_v\,\lambda_0}{s}\,\hat{\mathbf{z}} \qquad \gamma_v \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{12.10}$$

with v given as above.

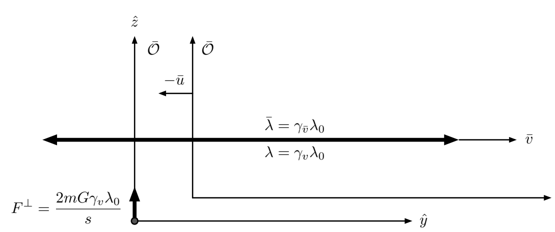

Figure 12.4: In the stationary frame of the test mass m (here called $\mathcal{O}$), the lab frame is moving to the left with speed $\bar{u}$. We can calculate the force in $\mathcal{O}$ and transform it to $\bar{\mathcal{O}}$.

Finally, in order to compare the force on the particle in $\mathcal{O}$ with the force on the particle in $\bar{\mathcal{O}}$, we must transform the perpendicular component of the force according $\bar{\mathbf{F}}^{\perp} = \gamma_{\bar{u}}^{-1}\,\mathbf{F}^{\perp}$. That is, we take the force in $\mathcal{O}$ and multiply it by (one over) the boost factor associated with the relative motion of $\mathcal{O}$ and $\bar{\mathcal{O}}$: $\gamma_{\bar{u}} \equiv \frac{1}{\sqrt{1-\frac{\bar{u}^2}{c^2}}}$. When we perform this transformation, we get the force perpendicular to the relative motion as measured in the lab frame $\bar{\mathcal{O}}$:

$$\bar{\mathbf{F}}^{\perp} = \frac{2\,m\,G\,\gamma_v\,\lambda_0}{\gamma_{\bar{u}}\,s}\,\hat{\mathbf{z}} \tag{12.11}$$

where I leave the perpendicular reminder to distinguish this result from the direct measurement of force in $\bar{\mathcal{O}}$ represented by (12.8).

12.2.3 Comparison

If Newtonian gravity supported the notion that inertial frames should make identical physical predictions, even if they disagree on phenomenology, then the force in (12.11) should be equal in magnitude and direction to (12.8). To see that these are not, in fact, equal, note that (in an uninteresting tour de force of algebraic investigation):

$$\frac{\gamma_v}{\gamma_{\bar{u}}} = \frac{c^2\left(1-\frac{\bar{u}\,\bar{v}}{c^2}\right)}{c^2\sqrt{1-\frac{\bar{v}^2}{c^2}}} = \gamma_{\bar{v}}\left(1-\frac{\bar{u}\,\bar{v}}{c^2}\right). \tag{12.12}$$

If we take this observation, and insert it into (12.11), then

$$\bar{\mathbf{F}}^{\perp} = \frac{2\,m\,G\,\gamma_{\bar{v}}\,\lambda_0}{s}\left(1-\frac{\bar{u}\,\bar{v}}{c^2}\right)\hat{\mathbf{z}}. \tag{12.13}$$

Compare this with the result (12.8):

$$\bar{\mathbf{F}} = \frac{2\,m\,G\,\gamma_{\bar{v}}\,\lambda_0}{s}\,\hat{\mathbf{z}} \tag{12.14}$$

and it is clear that the two frames do *not* agree on the force felt by the particle.

12.3 Electro-Magnetic Salvation

Let's briefly review how the dual electric and magnetic fields save the force predictions on the E&M side. The difference is in the fields associated with the $\bar{\mathcal{O}}$ frame shown in Figure 12.3 – if we now interpret λ_0 as a line charge, then we have both an electric and magnetic field in $\bar{\mathcal{O}}$, leading to both electric and magnetic forces – the fields at the particle location are:

$$\bar{\mathbf{E}} = -\frac{\gamma_{\bar{v}}\,\lambda_0}{2\,\pi\,\epsilon_0\,s}\,\hat{\mathbf{z}} \qquad \bar{\mathbf{B}} = -\frac{\mu_0\,\gamma_{\bar{v}}\,\lambda_0\,\bar{v}}{2\,\pi\,s}\,\hat{\mathbf{x}}, \tag{12.15}$$

and the force on a positive test charge is the usual $q\,\bar{\mathbf{E}} + q\,\bar{u}\,\hat{\mathbf{y}} \times \bar{\mathbf{B}}$:

$$\begin{aligned} \bar{\mathbf{F}}^{\perp} &= \left(-\frac{q\,\gamma_{\bar{v}}\,\lambda_0}{2\,\pi\,\epsilon_0\,s} + \frac{q\,\frac{\bar{u}\,\bar{v}}{c^2}\,\gamma_{\bar{v}}\,\lambda_0}{2\,\pi\,\epsilon_0\,s}\right)\hat{\mathbf{z}} \\ &= -\frac{q\,\gamma_{\bar{v}}\,\lambda_0}{2\,\pi\,\epsilon_0\,s}\left(1 - \frac{\bar{u}\,\bar{v}}{c^2}\right)\hat{\mathbf{z}}, \end{aligned} \tag{12.16}$$

transcribing the gravitational result in the particle rest frame (12.13), we have, on the electromagnetic side:

$$\bar{\mathbf{F}}^{\perp} = -\frac{q\,\gamma_{\bar{v}}}{2\,\pi\,\epsilon_0\,s}\left(1 - \frac{\bar{u}\,\bar{v}}{c^2}\right)\hat{\mathbf{z}}, \tag{12.17}$$

and, as expected, the physical predictions agree in this case.

12.4 Conclusion

The difference between this Newtonian gravitational argument and the same problem analyzed for line charges is that a moving line charge generates a "magnetostatic" force in the lab frame that is precisely the additional component found in (12.13) (i.e. in the electromagnetic case, the moving line of charge in the lab generates an electric and magnetic force on the test particle). So in E&M, the forces are identical, and the physical predictions in the two frames coincide.

The existence of a magnetic force solves the "problem", and it is tempting to put precisely an analagous "moving mass" source into Newtonian gravity. That is, we augment Newtonian gravity's electrostatic analogy with a similar magnetostatic term. This additional force is known as the "gravitomagnetic" force, and exists in almost the form implied by E&M as a linearized limit of General Relativity. There are quantitative differences beyond the obvious replacement of signs and units associated with the linearized GR limit, but the qualitative predictions that can be made by analogy with E&M provide excellent "intuition" into weak gravitational problems.

As an example, consider a spinning massive sphere with uniform charge density and constant angular velocity $\vec{\omega} = \omega\,\hat{\mathbf{z}}$. On the electromagnetic side, we expect the exterior electric and magnetic fields to look like Coulomb and a magnetic dipole respectively. If we took a spinning, charged test body, we would expect the spin of the test body to precess in the presence of the magnetic dipole field (Larmor precession is an example).

In the weak-field GR limit, we expect a replacement similar to (12.5) to apply to both the electric and magnetic fields, giving us a Newtonian and gravitomagnetic contribution. If we then introduced a spinning test mass, we would expect precession just as in the electromagnetic case. In this gravitational setting, such precession is referred to as Lense-Thirring precession, and the Gravity Probe B experiment is currently measuring this effect in an orbit about the earth (a spinning, massive ball).