Relativistic Solutions

Lecture 11

Physics 411
Classical Mechanics II

September 21st, 2007

With our relativistic equations of motion, we can study the solutions for $x(t)$ under a variety of different forces. The hallmark of a relativistic solution, as compared with a classical one, is the bound on velocity for massive particles. We shall see this in the context of a constant force, a spring force, and a one-dimensional Coulomb force.

This tour of potential interactions leads us to the question of what types of force are even allowed – we will see changes in the dynamics of a particle, but what about the relativistic viability of the mechanism causing the forces? A real spring, for example, would break long before a mass on the end of it was accelerated to speeds nearing c. So we are thinking of the spring force, for example, as an approximation to a well in a more realistic potential. To the extent that effective potentials are uninteresting (or even allowed in general), we really only have one classical, relativistic force – the Lorentz force of electrodynamics.

11.1 Free Particle Motion

For the equations of motion in proper time parametrization, we have

$$\ddot{x}^\mu = 0 \longrightarrow x^\mu = A^\mu \tau + B^\mu. \tag{11.1}$$

Suppose we rewrite this solution in terms of t – inverting the $x^0 = c\,t$ equation with the initial conditions that $t(\tau = 0) = 0$, we have

$$\tau = \frac{c\,t}{A^0} \tag{11.2}$$

and then the rest of the equations read:

$$x = A^1 \frac{c\,t}{A^0} + B^1 \quad y = A^2 \frac{c\,t}{A^0} + B^2 \quad z = A^3 \frac{c\,t}{A^0} + B^3, \tag{11.3}$$

and we can identify the velocities:

$$v^x = \frac{A^1\,c}{A^0} \qquad v^y = \frac{A^2\,c}{A^0} \qquad v^z = \frac{A^3\,c}{A^0}. \tag{11.4}$$

Remember that we always have the constraint that defines proper time:

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{11.5}$$

and this can be used to relate A^0 to the rest of the constants:

$$A^0 = \frac{c}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{11.6}$$

The "lab" measurements, which proceed with a clock reading coordinate time, give back constant velocity motion with velocity $\mathbf{v}$ and we pick up a relation between the coordinate time and the proper time of the particle (in its own rest frame).

11.2 Motion under a Constant Force

Constant force, according to Newton's second law, is one for which $\frac{dp}{dt} = F$ with constant F. In one spatial dimension, we can describe relativistic constant forcing by interpreting p as the spatial, *relativistic* momentum, i.e. $p = \frac{m\,v}{\sqrt{1-v^2/c^2}}$. From this point of view, the solution is straightforward – to avoid confusion, let $v(t) = x'(t) \equiv \frac{dx(t)}{dt}$, then

$$\frac{d}{dt}\left(\frac{m\,x'(t)}{\sqrt{1-x'(t)^2/c^2}}\right) = F \longrightarrow x'(t) = \frac{F\,t}{m\sqrt{1+\frac{F^2\,t^2}{m^2\,c^2}}} \tag{11.7}$$

which can be integrated again – if we set $x(t=0)=0$, then

$$x(t) = \frac{m\,c^2}{F}\left(\sqrt{1+\left(\frac{F\,t}{m\,c}\right)^2} - 1\right), \tag{11.8}$$

the usual hyperbolic motion.

If we start with the relativistic free particle Lagrangian in proper time parametrization, $L = -m\,c\,\sqrt{-\dot{x}^\mu\,\eta_{\mu\nu}\,\dot{x}^\nu}$, then we'd like to add a "potential" that gives us the constant force case. We know such a potential will be . . . strange – after all, it must be a scalar, meaning here that it will have to be constructed out of vectors (specifically, combinations of $x^\mu(\tau)$ and its τ derivatives). In this two-dimensional setting, about the only thing that suggests itself is the cross product of x^μ and $\dot{x}^\mu$. As we shall see, the Lagrangian:

$$L = -m\,c\,\sqrt{-\dot{x}^\mu\,\eta_{\mu\nu}\,\dot{x}^\nu} + \frac{1}{2}\,\frac{F}{c}\,\epsilon_{\mu\nu}\,\dot{x}^\mu\,x^\nu \tag{11.9}$$

corresponds to the constant force as defined above. Let's work out the equations of motion – written out, the Lagrangian is

$$L = -m\,c\,\sqrt{c^2\,\dot{t}^2 - \dot{x}^2} + F(x\,\dot{t} - t\,\dot{x}). \tag{11.10}$$

Remember, from the definition of proper time, we have $c^2 = c^2\,\dot{t}^2 - \dot{x}^2$, which we can solve for $\dot{x}$, for example – then

$$\dot{x} = c\,\sqrt{\dot{t}^2 - 1} \qquad \ddot{x} = \frac{c\,\dot{t}\,\ddot{t}}{\sqrt{\dot{t}^2 - 1}}. \tag{11.11}$$

Using the proper time relation, we get equations of motion as follows:

$$\begin{aligned} \frac{d}{d\tau}\,\frac{\partial L}{\partial \dot{t}} - \frac{\partial L}{\partial t} = 0 \longrightarrow m\,c\,\ddot{t} = \frac{F}{c}\,\dot{x} \\ \frac{d}{d\tau}\,\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \longrightarrow m\,\ddot{x} = F\,\dot{t}. \end{aligned} \tag{11.12}$$

Upon introducing the equalities (11.11), these degenerate to the single equation:

$$\ddot{t} = \frac{F}{m\,c}\sqrt{\dot{t}^2 - 1} \tag{11.13}$$

with solution:

$$t(\tau) = \alpha + \frac{m\,c}{F}\left(\cosh(\beta)\,\sinh\left(\frac{F\,\tau}{m\,c}\right) + \sinh(\beta)\,\cosh\left(\frac{F\,\tau}{m\,c}\right)\right), \tag{11.14}$$

setting $\alpha = \beta = 0$ (our choice) to get $t(\tau = 0) = 0$, we have

$$t(\tau) = \frac{m\,c}{F}\,\sinh\left(\frac{F\,\tau}{m\,c}\right) \longrightarrow \tau(t) = \frac{m\,c}{F}\,\sinh^{-1}\left(\frac{F\,t}{m\,c}\right). \tag{11.15}$$

Meanwhile, we can solve $\dot{x}$ from (11.11):

$$x(\tau) = \alpha + \frac{m\,c^2}{F}\cosh\left(\frac{F\,\tau}{m\,c}\right), \tag{11.16}$$

this is the spatial coordinate in proper time parametrization – to find $x(t)$, we can use the inverse relation $\tau(t)$ and just replace:

$$x(t) = \alpha + \frac{m\,c^2}{F}\sqrt{1 + \frac{F^2\,t^2}{m^2\,c^2}} \tag{11.17}$$

and our lone integration constant α can be used to set $x(t=0)=0$, the physical boundary condition. When all is said and done, we recover (11.8).

11.3 Motion Under Hookean Potential

We can also consider a relativistic spring – here the difficulty is that for an arbitrary displacement, the maximum velocity can be arbitrarily large classically. We expect our relativistic mechanics to take care of this, providing a maximum speed $< c$. This time, we will start from the relativistic Lagrangian written in coordinate time parametrization:

$$L = -m\,c^2\sqrt{1 - \frac{x'(t)^2}{c^2}} - \frac{1}{2}\,k\,x(t)^2 \tag{11.18}$$

and the potential is the usual one with spring constant k. As a check, this gives the correct Newton's second law form upon variation:

$$\frac{d}{dt}\frac{\partial L}{\partial x'(t)} - \frac{\partial L}{\partial x(t)} = 0 \longrightarrow \frac{d}{dt}\left(\frac{m\,x'(t)}{\sqrt{1 - \frac{x'(t)^2}{c^2}}}\right) = -k\,x(t). \tag{11.19}$$

We could integrate this directly (not easy). Instead, we can use the Hamiltonian associated with L:

$$\begin{aligned} H &= \frac{\partial L}{\partial x'(t)}\,x'(t) - L = \frac{m\,x'(t)^2}{\sqrt{1 - \frac{x'(t)^2}{c^2}}} - \left(-m\,c^2\sqrt{1 - \frac{x'(t)^2}{c^2}} - \frac{1}{2}\,k\,x(t)^2\right) \\ &= \frac{m\,c^2}{\sqrt{1 - \frac{x'(t)^2}{c^2}}} + \frac{1}{2}\,k\,x(t)^2, \end{aligned} \tag{11.20}$$

no surprise. But it does tell us that numerically, this particular combination is a constant, we'll call it the total energy (notice that it contains the rest energy as well as the kinetic and potential energies). Setting $H = E$ gives us a first integral of the motion, and we could solve it[1] – instead, we can take the total time derivative to recover a second order ODE analagous to the equation of motion itself:

$$\frac{m\,x''(t)}{\left(1 - \frac{x'(t)^2}{c^2}\right)^{3/2}} = -k\,x(t). \tag{11.21}$$

From here, it is easiest to find the solution numerically – we'll use initial conditions $x(t=0) = x_0$, $x'(t=0) = 0$, so we're starting at maximum amplitude, then the position and velocity for a few different starting locations are shown in Figure 11.1 and Figure 11.2. Notice that the "low-velocity" case, corresponding to the smallest starting position looks effectively just like a classical spring, both in position and velocity. As the initial amplitude is increased, the solution for $x(t)$ begins to look more like a sawtooth as the velocity turns into a step function close to c.

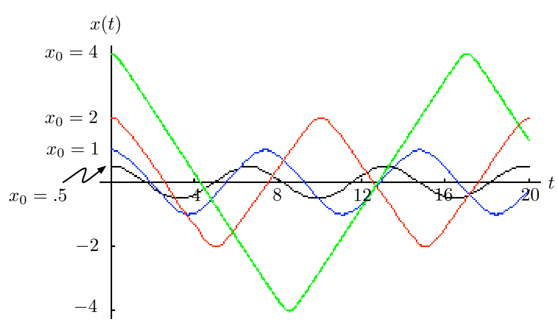

Figure 11.1: Position for a relativistic spring (this is the result of numerical integration, with $k = 1$, $c = 1$ and x_0 shown).

[1] To get the asymptotic behavior, solve $H = E$ for v in terms of x and send $E \to \infty$ – the relation you will recover is $v = \pm c$, i.e. the sawtooth motion is predictable from the Hamiltonian

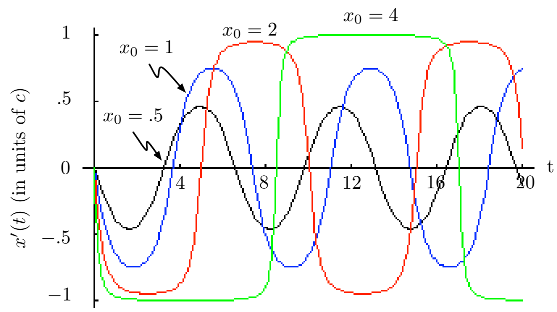

Figure 11.2: Velocity for the relativistic spring potential (here, $k = 1$, $c = 1$, and we start at a variety of maximum amplitudes as shown).

11.4 Infall

Consider the one-dimensional infall problem for a test charge moving under the Coulomb field coming from a point charge at the origin. The total energy of the test charge is

$$E = \frac{1}{2} m \dot{x}^2 - \frac{q^2}{x}. \tag{11.22}$$

If we start a particle from rest at spatial infinity at time $t = -\infty$, then the energy is $E = 0$, and this is conserved along the trajectory. We can easily solve the above with $E = 0$ and integration constant written in terms of t^*, the time it takes to reach the origin (starting from $t = 0$):

$$x(t) = \left(\frac{3}{2}\right)^{2/3} \left(\sqrt{\frac{2}{m}}\, q\, (t^* - t)\right)^{2/3}. \tag{11.23}$$

The problem, as always, with this classical solution is that the velocity grows arbitrarily large as $t \longrightarrow t^*$:

$$v(t) = \frac{dx}{dt} \sim (t^* - t)^{-1/3}, \tag{11.24}$$

allowing the test particle to move faster than light.

Once again, the relativistic Lagrangian, written in temporal parametrization, provides dynamics with a cut-off velocity. The relativistic Hamiltonian (total energy) reads

$$E = \frac{m c^2}{\sqrt{1 - \frac{x'(t)^2}{c^2}}} - \frac{q^2}{x}. \tag{11.25}$$

Now for a particle starting at rest at spatial infinity, we have $E = m\,c^2$, not the classical zero. Once again, we can solve the above with $E = m\,c^2$ to get the dynamics, The full form of the solution is not particularly enlightening – what is important, and different from the classical case, is the behavior of the velocity at t^*: For our relativistic solution, the velocity at the origin is $-c$. In Figure 11.3, we have the relativistic (black lines) and classical (blue lines) position and velocities for the same t^*.

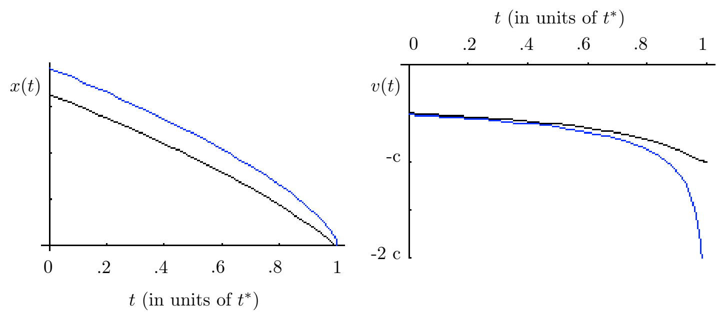

Figure 11.3: Position (left) and velocity (right) for $t = 0 \rightarrow t^*$ for classical (blue) and relativistic (black) infall trajectories.

11.5 From Whence, The Force?

We have just covered some of the canonical set of forces that can be studied in relatively closed form relativistically. But more than that, there is the question of what forces are even *consistent* with special relativity, let alone nicely solvable. Thus begins our journey – the only force entirely in accord with the demands of special relativity is the Lorentz force of electrodynamics. As the theory which, to a certain extent, inspired Einstein towards a spacetime view, it is not surprising that the Lorentz force is the one (and only, on the classical side) that provides a consistent description.

There are two immediate requirements for special relativity when it comes to a force associated with a field theory. The first is that the force be robust enough to offer the same prediction in two different inertial frames. The second (related) is that the mediator of the force (to wit: the field) respond to changes in its source with finite propagation speed (a causality issue). These are just rewrites of the usual two postulate approach to special relativity – equivalence of inertial frames and finite information propagation.

This is the source of the original issue that would appear to be at the heart of this course: General relativity is a theory of gravity. What's wrong with the Newtonian theory? As we shall see next time, Newtonian gravity fails both postulates of special relativity. Given the success of electrodynamics (passes both), it is tempting to construct an analagous gravitational theory, and such a linear vector theory is successful up to a point, but cannot address the unique qualities of gravity as an observational phenomenon: Everything is influenced by gravity, and the gravitational "force" between two objects is attractive (only one flavor of "charge").