Radial Equation

Lecture 21

Physics 342
Quantum Mechanics I

Wednesday, March 26th, 2008

Today we'll tie up some loose ends and finish our discussion of the radial equation. Much of the material discussed in class is in previous notes.

Homework

Reading: Griffiths, pp. 140–145.

Problem 21.1

It is often useful to obtain an energy discretization relation without constructing the full (radial) wavefunction. We have been associating discrete spectra with boundary conditions – either explicit (like the infinite square well) or implicit (like the requirement of normalizability from the harmonic oscillator). In this problem, we will get the spectrum of the three-dimensional harmonic oscillator, without worrying about the wavefunctions themselves.

a. Write the three-dimensional harmonic oscillator potential as $V(r) = \frac{1}{2}\, m\, \omega\, r^2$, and insert this into the one-dimensional radial equation with effective potential built-in (remember that $u(r) = r\, f(r)$ where $f(r)$ is the actual radial portion of $\psi(r, \theta, \phi) = f(r)\, g(\theta)\, h(\phi)$):

$$-\frac{\hbar^2}{2\, m}\, u''(r) + \left[V(r) + \frac{\hbar^2}{2\, m}\, \frac{\ell\,(\ell+1)}{r^2}\right] u(r) = E\, u(r). \tag{21.1}$$

Identify a constant A (that depends on ω, among other things) with units of L^{-1}, and use this to rewrite the above in terms of $z = A\, r$, i.e. find the unitless form.

b. As with the one-dimensional harmonic oscillator, $u(z) \sim e^{\pm\frac{1}{2}z^2}$ for large z – rewrite your equation from part a. in terms of $\bar{u}(z)$ defined by

$$u(z) = e^{-\frac{1}{2}z^2}\,\bar{u}(z). \tag{21.2}$$

c. Set

$$\bar{u} = z^p \sum_{j=0}^{\infty} \alpha_j\, z^j \tag{21.3}$$

and insert into your ODE from part b. Solve the indicial equation, write the recursion relation, assume the series truncates at some value (this is the same as the argument for the one-dimensional case) $j = J$ and find E_J.

Problem 21.2

The three-dimensional harmonic oscillator potential can also be solved using Cartesian separation – write the time-independent Schrödinger equation in three dimensional Cartesian coordinates for $V(r) = \frac{1}{2}\,m\,\omega^2\,r^2$. Now argue that you have three one-dimensional oscillators, and use this to find the energy E associated with the three-dimensional oscillator. You should have three constants of integration in your solution. Does your solution make sense when compared to the energy discretization you got in Problem 21.1?