

Laboratory III: Operational Amplifiers

1 Introduction

Operational amplifiers are one of the most useful building blocks of analog electronics. Ideally, an op amp would have an infinite gain, infinite bandwidth, infinite input resistance, and zero output resistance. The functioning of an ideal op amp can be understood by the “golden rules”, which are

GRI: there is no voltage difference between the inputs

GRII: the inputs draw no current.

In this week’s lab we will investigate several circuits in order to understand the utility as well as the limitations of real-world op-amps.

The finite input resistance of real-world op-amps means that op amps can potentially “load down” the previous stage in a circuit, but this is very rare. More problematic is the input voltage offset V_{OS} , because if the op amp is used as an amplifier with large gain the nominally small V_{OS} will also be amplified. The most serious limitations of opamps however have to do with their response time. If the frequency of the input signal is too high, the op amp will not work as expected. For a detailed discussion of op amp limitations please read Simpson [2] chapter 9.6 - 9.8. The purpose of this hand out is to illuminate some points that Simpson does not discuss in detail.

Before we get to that, let us define some terms that will be used throughout.

2 Jargon

Rolloff Decrease of gain as a function of frequency in the stopband region.

dB (decibel) A ratio of powers can be expressed in decibel by computing

$$\text{power ratio in decibel} = 10 \log_{10} \left(\frac{P}{P_0} \right) \quad (1)$$

Where P is the measured power of interest and P_0 is a common reference power that has to be specified. Since power is proportional to the square of the voltage $P \propto V^2$, a voltage ratio in decibels is computed as follows

$$\text{voltage ratio in decibel} = 10 \log_{10} \left(\frac{V^2}{V_0^2} \right) = 10 \log_{10} \left(\left[\frac{V}{V_0} \right]^2 \right) = 20 \log_{10} \left(\frac{V}{V_0} \right) \quad (2)$$

A voltage gain is nothing but a voltage ratio. A factor of $\sqrt{2}$ decrease in voltage (factor 2 in power), meaning the output voltage has an amplitude that is $1/\sqrt{2}$ times the amplitude of the input voltage, $V_{out}/V_{in} = 1/\sqrt{2}$, implies a gain of -3dB, because $\text{dB} = 20 \log_{10}(1/\sqrt{2}) = -3.01$. If $V_{out}/V_{in} = 1/2$, then $\text{dB} = 20 \log(1/2) = -6.02$. If $V_{out}/V_{in} = 1/10$, then $\text{dB} = 20 \log(1/10) = -20$. If $V_{out}/V_{in} = 1/100$, then $\text{dB} = 20 \log(1/100) = -40$.

20 dB per decade A function $V(f)$ which is proportional to $1/f$ is said to “fall off” (or “roll off”) at the rate of 20 dB per decade. That is, for every factor of 10 in frequency f (every “decade”), the amplitude drops 20 dB because

$$20 \log_{10} \left(\frac{V(10f)}{V(f)} \right) = 20 \log_{10} \left(\frac{1/(10f)}{1/f} \right) = 20 \log_{10} \left(\frac{1}{10} \right) = -20 \quad (3)$$

6 dB per octave A function $V(f)$ which is proportional to $1/f$ is said to fall off 6 dB per octave. That is, for every factor of 2 in the frequency f (every “octave”), the voltage drops close to 6 dB. Thus, 6 dB per octave is the same thing as 20 dB per decade.

H - the transfer function. Although Simpson uses a different notation, the standard notation for the transfer function is H . A given $H(\omega)$ fully characterizes the gain and phase of a linear filter. It is a frequency dependent function. In our context

$$H(\omega) = \frac{v_{out}}{v_{in}}. \quad (4)$$

where $v_x = V_x e^{i\varphi_x} e^{i\omega t}$. The filter gain at a given frequency is simply the magnitude of the transfer function

$$\text{gain at } \omega = |H(\omega)| = \frac{|v_{out}|}{|v_{in}|} = \frac{V_{out}}{V_{in}}. \quad (5)$$

Slew Rate The output slew-rate of an amplifier or other electronic circuits is defined as the maximum rate of change of the output voltage for all possible input signals.

$$\text{SR} = \max \left(\left| \frac{dv_{out}(t)}{dt} \right| \right) \quad (6)$$

where $v_{out}(t)$ is the output produced by the amplifier as a function of time t .

Rise Time is used to quantify the step response of a system, *i.e.* to quantify the time evolution a circuit’s voltage (or current) output when its input is a Heaviside step function. One measures the rise-time by determining the time required for a signal to change from a specified low value to a specified high value. Typically, these values are 10% and 90% of the step height.

3 First-Order Low-Pass Filters

The operational amplifiers we are using are internally compensated, which means that they are stable and will not self-oscillate because an internal capacitor forces the open loop gain to roll-off by 6 dB/octave (20 dB/decade) down to unity gain (Simpson pg. 388). The behavior of the op amp gain as a function of frequency is in this case the same as that of a simple low-pass RC-filter. Therefore, let us review low pass filters.

You probably remember from Physics 200 that $2\pi f_{3dB} = 1/\tau = 1/(RC)$ for a simple RC low pass filter. I want to show this again looking at the problem from several angles. As a benefit we will derive Eq. (9.17) in Simpson pg. 384, which relates the rise time to the bandwidth.

3.1 Differential Equation

For the RC circuit depicted in Fig. 1 we may use Kirchoff's laws

$$\begin{aligned}\frac{v_{in} - v_{out}}{R} &= I \\ \frac{dv_{out}}{dt} &= \frac{I}{C}\end{aligned}$$

to derive the following equation relating the input signal $v_{in}(t)$ to the output voltage $v_{out}(t)$

$$RC \frac{dv_{out}}{dt} + v_{out} = v_{in}. \quad (7)$$

3.2 Frequency Domain

3.2.1 The Transfer Function $H(\omega)$

The transfer function H fully specifies the response of any linear circuit. H is a function of frequency and can be understood as characterizing for each frequency the circuit response to a sinusoidal input, *i.e.* the signal gain and phase shift of the circuit.

(A) We will derive the low-pass filter transfer function $H(\omega)$ from Eq. (7). To do so consider the filter response to a sin-wave. Because for linear circuits the input and output frequency are always equal we may use the ansatz

$$v_{in}(t) = V_{in} e^{i\omega t}, \quad v_{out}(t) = V_{out} e^{i\phi} e^{i\omega t}. \quad (8)$$

Plug this into Eq. (7) to obtain

$$H(\omega) = \frac{v_{out}(t)}{v_{in}(t)} = \frac{1}{i\omega RC + 1} \quad (9)$$

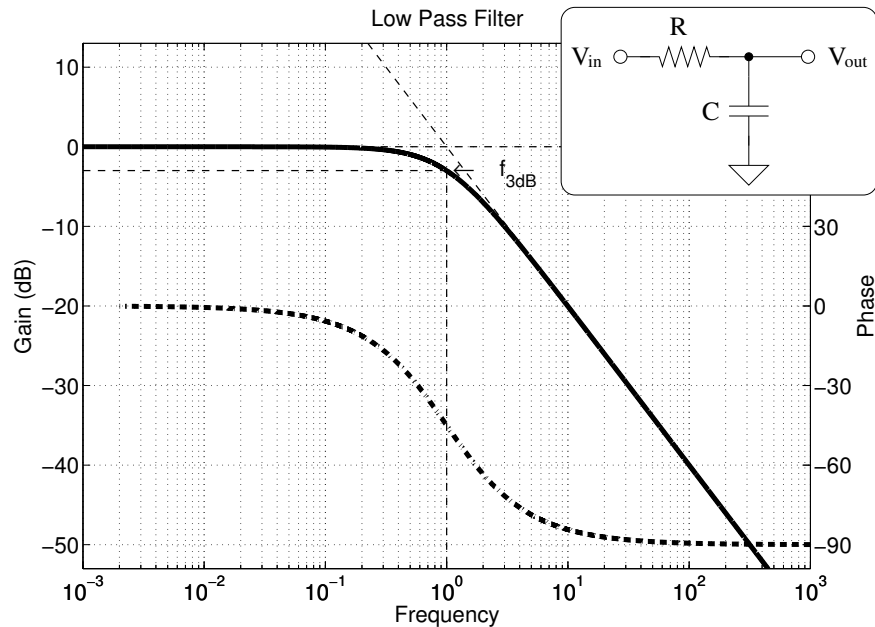


Figure 1: RC low pass filter.

(B) We will derive, just for fun, the transfer function using complex impedances. To do so notice that the circuit shown in Fig. 1 may be viewed as a simple voltage divider with

$$Z_1 = R \quad Z_2 = \frac{1}{i\omega C} \quad (10)$$

and therefore

$$H(\omega) = \frac{v_{out}(t)}{v_{in}(t)} = \frac{Z_2}{Z_1 + Z_2} = \frac{1}{i\omega C \left(R + \frac{1}{i\omega C} \right)} = \frac{1}{i\omega RC + 1} \quad (11)$$

It is useful to rewrite the transfer function as

$$H(\omega) = \frac{1}{i\omega RC + 1} = \frac{1}{\sqrt{(\omega RC)^2 + 1}} e^{i \arctan(\omega RC)}, \quad (12)$$

where the first term is the voltage-gain ($|H|$) and the second term specifies the induced phase-shift. The low-pass gain and phase-shift are shown in Fig. 1.

3.2.2 Bandwidth and Corner Frequency f_{3dB}

A low pass filter has a constant gain at low frequencies and rolls off for higher frequencies, as can be seen from the plot of the gain ($= |H|$) in Fig. 1. The transition from constant to roll-off is quantified by the 3 dB frequency, the frequency for which the gain $|H|$ has fallen by

a factor of $\sqrt{2}$ from its maximum. For low-pass filters the gain is maximum at zero frequency. For the passive RC-low pass filter

$$g_{max} = \lim_{\omega \rightarrow 0} |H(\omega)| = \lim_{\omega \rightarrow 0} \frac{1}{\sqrt{(\omega RC)^2 + 1}} = 1 \quad (13)$$

as one would expect. Therefore, setting

$$\frac{|H(\omega_{3dB})|}{g_{max}} = \frac{1}{\sqrt{2}} = \left| \frac{1}{i\omega_{3dB}RC + 1} \right| = \frac{1}{\sqrt{(\omega_{3dB}RC)^2 + 1}} \quad (14)$$

we obtain for the corner frequency the familiar

$$2\pi f_{3dB} = \frac{1}{RC} \quad \Leftrightarrow \quad f_{3dB} = \frac{1}{2\pi RC}. \quad (15)$$

Because all frequencies below f_{3dB} are passed whereas signals with higher frequencies are suppressed, the frequency range $0 - f_{3dB}$ is called the *bandwidth* of the low-pass filter.

3.2.3 Roll-Off and Gain-Bandwidth Product

Frequencies below f_{3dB} are passed whereas signals with higher frequencies are suppressed:

$$\begin{aligned} |H(f)| &\approx |H(0)| = 1, & \text{if } f < f_{3dB}, \\ |H(f)| &\approx \frac{1}{2\pi RC f} = \frac{f_{3dB}}{f}, & \text{if } f > f_{3dB}. \end{aligned}$$

The gain “roll-off” (jargon for the decrease with respect to frequency) beyond the -3 dB point is “-20 dB per decade”, where decade refers to a factor of ten in frequency and -20 dB/decade indicates a ten-fold decrease in gain for a ten-fold increase in frequency.

The same in not so many words: For $f_2 = 10 \times f_1$, it holds that

$$20 \log_{10} [|H(f_2)|] - 20 \log_{10} [|H(f_1)|] = 20 \log_{10} \left[\frac{|H(f_2)|}{|H(f_1)|} \right] = 20 \log_{10} \left[\frac{1}{10} \right] = -20.$$

Because the gain roll-off is proportional to f^{-1} , it is useful to introduce the *gain-bandwidth product* (GBW) because it is a constant ($=f_{3dB}$ for the RC low-pass filter):

$$GBW = |H(f)| f = |H(0)| f_{3dB} = \text{const.} \quad (f > f_{3dB}) \quad (16)$$

Knowledge of this constant is all we need in order to figure out the circuit gain for frequencies beyond f_{3dB} .

3.3 Time Domain:

Instead of probing a circuit with a series of sinusoidal signals for all frequencies of interest, one might input a step function (or very slow square wave) and look at the response. In this case many frequencies are probed at once (instead of just one frequency for a sine-wave) but to obtain useful information we must measure the full time-evolution of the resulting output (instead of just its amplitude). That is, we analyze the circuit in the time-domain. To describe this situation theoretically, we need to solve Eq. (7) for a step-input.

3.3.1 Solution of Eq. (7)

Before considering the response to a step function let us integrate Eq. (7) for general input signals. To do so, it is useful to first multiply Eq. (7) by an appropriate factor in order to obtain a form that can be integrated by inspection. Consider the factor

$$\alpha(t) = e^{\int 1/(RC)dt} = e^{\frac{t}{RC}}. \quad (17)$$

then

$$\begin{aligned} \frac{d}{dt}(\alpha v_{out}) &= e^{\frac{t}{RC}} \frac{dv_{out}}{dt} + (RC)^{-1} e^{\frac{t}{RC}} v_{out} \\ &= (RC)^{-1} e^{\frac{t}{RC}} \left((RC) \frac{dv_{out}}{dt} + v_{out} \right) \\ &\stackrel{\text{Eq. (7)}}{=} (RC)^{-1} e^{\frac{t}{RC}} v_{in} \end{aligned} \quad (18)$$

And the solution is then obtained in a straightforward manner from

$$\alpha(t)v_{out}(t) - \alpha(t_0)v_{out}(t_0) = \int_{t_0}^t (RC)^{-1} e^{\frac{s}{RC}} v_{in} ds, \quad (19)$$

yielding

$$v_{out}(t) = v_{out}(t_0) e^{-\frac{(t-t_0)}{RC}} + e^{-\frac{t}{RC}} \int_{t_0}^t (RC)^{-1} e^{\frac{s}{RC}} v_{in} ds \quad (20)$$

The second term in in Eq. (20) is called the particular solution and the first term is the solution of the homogeneous equation, the equation with $v_{in} = 0$. It is seen that the homogeneous solution-term decays exponentially, which is why it is often referred to as the transient term. Measurements are typically taken after sufficient time has passed such that transient oscillations have died down and circuit behavior is only a function of the input and does not anymore depend on the unknown initial voltage, which for this circuit is caused by the unknown initial charge on the capacitor. Mathematically this is achieved by using the limit $t_0 \rightarrow -\infty$

$$v_{out}(t) = e^{-\frac{t}{RC}} \int_{-\infty}^t (RC)^{-1} e^{\frac{s}{RC}} v_{in} ds \quad (21)$$

Returning to the question at hand, the response to a unit-step input, $v_{in} = \theta(t)$:

$$v_{out}(t) = e^{\frac{-t}{RC}} \int_{-\infty}^t (RC)^{-1} e^{\frac{s}{RC}} \theta(t) ds$$

Clearly $v_{out} = 0$ for $t < 0$ and for $t > 0$

$$\begin{aligned} v_{out}(t) &= e^{\frac{-t}{RC}} \int_0^t (RC)^{-1} e^{\frac{s}{RC}} ds \\ &= e^{\frac{-t}{RC}} \left(e^{\frac{t}{RC}} - 1 \right) \\ &= 1 - e^{\frac{-t}{RC}}. \end{aligned}$$

In other words

$$v_{out}(t) = \theta(t) (1 - e^{-t/\tau_{RC}}), \quad (22)$$

where we have introduced the decay time

$$\tau_{RC} = RC. \quad (23)$$

For later use, we will introduce a second notation for this time that more directly reflects the way it would be measured

$$t_{63\%} = \tau_{RC}. \quad (24)$$

Here $t_{63\%}$ is defined as the time it takes the voltage to reach $1 - e^{-1} = 0.63$ of its final value, which in this case is 1.

3.3.2 Risetime

The risetime (R.T.) is defined as the time it takes a signal to go between 10% and 90% of its peak-to-peak voltage. Let's calculate this time from Eq. (22) using that v_{out} is initially 0 and tends to 1 for large t .

$$\begin{aligned} v_{out}(t_{90\%}) &= 0.9 = (1 - e^{-t_{90\%}/\tau_{RC}}) \\ \Leftrightarrow e^{-t_{90\%}/\tau_{RC}} &= 0.1 \\ \Leftrightarrow t_{90\%} &= -\ln(0.1) \tau_{RC} \end{aligned}$$

Similarly

$$t_{10\%} = -\ln(0.9) \tau_{RC}$$

yielding

$$R.T. = t_{90\%} - t_{10\%} = [-\ln(0.1) + \ln(0.9)] \tau_{RC} = \ln 9 \tau_{RC} = 2.2 \tau_{RC} \quad (25)$$

3.4 RC low-pass summary

From Eq. (15), Eq. (16), Eq. (23), Eq. (24), and Eq. (25)

$$GBW = |H(f)| f = |H(0)| f_{3dB} = f_{3dB} \quad (f > f_{3dB}) \quad (26)$$

$$t_{63\%} = \tau_{RC} = RC = \frac{1}{2\pi f_{3dB}} \quad (27)$$

$$\text{R.T.} = \frac{\ln 9}{2\pi} \frac{1}{f_{3dB}} = \frac{0.35}{f_{3dB}} \quad (28)$$

The last equality is precisely Eq. (9.17) in Simpson pg. 384 and relates the rise time, which is obtained through a time-domain measurement, to the bandwidth (f_{3dB}), which is obtained through a frequency-domain measurement.

This relation demonstrates the important fact that a limited circuit bandwidth always implies a limit on the possible rise-time of the circuit, be it a RC filter or an op-amp.

4 What limits the performance: the gain bandwidth product or the slew rate ?

The gain-bandwidth product of an operational amplifier is given by the product of the DC gain of the amplifier and the frequency bandwidth (*i. e.* the -3 dB frequency f_{3dB}). This product is a meaningful figure because internally compensated op-amps, such as the 411, have an amplitude-frequency response of a low-pass filter. As a result, the gain “roll-off” (jargon for the decrease with respect to frequency) beyond the -3 dB point is “-20 dB per decade”.

To know the GBW of an opamp you need to consult the datasheet or attempt to measure the roll-off of the huge *open loop* gain. For normal operation of opamps a negative feedback is used. (See Simpson [2] page 382, Fig. 9.11 for a comparison of the open and closed loop case). In this case there is a tradeoff between gain and bandwidth. A larger gain implies a lower f_{3dB} and a larger bandwidth (large f_{3dB}) is achieved by decreasing the gain $|H(0)|$.

The slew rate can also result in limited gain. In Simpson on pp. 389-390, it is shown that to faithfully produce a sine wave of frequency f (Hz) and amplitude V_0 (V) requires the amplifier’s slew rate to be at least $2\pi f V_0$, *i.e.* the threshold frequency for the roll-off region is given by

$$f_T = \frac{\text{S.R.}}{2\pi V_0}.$$

Note, that here the bandwidth depends on the signal amplitude! The limited slew rate also implies that for sine-waves with frequencies larger than f_T , the maximum peak-to-peak voltage ($V_{out,pp} = 2V_0$) is roughly [1],

$$V_{out,pp} \approx \frac{\text{S.R.}}{\pi f}.$$

Input	Feature	Gain bandwidth product	Slew Rate
sin	Rolloff	$ H = \frac{GBW}{f} \quad f > f_{3dB}$	$\text{gain} = \frac{S.R.}{\pi f V_{in,pp}} \quad f > f_T$
sin	Corner freq.	$f_{3dB} = \frac{GBW}{ H(0) }$	$f_T = \frac{S.R.}{\pi V_{out,pp}}$
step	Rise-time	$R.T. = \frac{0.35}{f_{3dB}}$	$R.T. = \frac{0.8 V_{out,pp}}{S.R.}$

Table 1: Comparison of op-amp performance limits due to limited slew rate and finite gain bandwidth product.

It is seen that the gain decreases with increasing frequency (-20 dB/decade) in the roll-off region because $g = V_{out,pp}/V_{in,pp} \approx (S.R.)/(V_{in,pp} \pi f) = f_T/f$.

For a step function input, the limited slew rate leads, in the simplest case, to a constant-slope output signal with a slope equal to the slew rate. Then the rise time is simply $R.T. = (S.R.)^{-1} 0.8 V_{out,pp}$.

In summary, both the finite gain-bandwidth product and the finite slew rate can lead to a finite rise-time and an op-amp gain that decreases with increasing frequency. In this lab you are asked to explore this.

5 The briefest review of error propagation

The experiments in Jlab are not meant to be precision measurements, a 10% relative measurement uncertainty is typically sufficient. Nevertheless, you need to be able to estimate the uncertainty of your final measurement result.

Error propagation considers the effect of variables' uncertainties (or errors) on the uncertainty of a function based on them. To make this discussion brief and simple, we will assume, and this is a big assumption, that all variables in question are *uncorrelated*. This assumption makes life simple because we do not have to consider the covariance but it may not always be a justifiable assumption. When thinking about errors, we will usually implicitly assume that the values of the variables follow a Gaussian probability distribution. In this case, one typically quantifies the uncertainty of a variable by the standard deviation, σ , where the standard deviation is the positive square root of the variance, σ^2 . In this context, specifying a variable value \pm one standard deviation ($x \pm \sigma$) means that there is a 68% probability that the true value lies in the region $x \pm \sigma$.

With these assumptions in mind, consider the simplest case of two uncorrelated variables x and y with uncertainties σ_x and σ_y and a function $f(x, y)$. What is σ_f ?

$$\sigma_f = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2} \quad (29)$$

That's all.

Example 1

$$f(x, y) = x \pm y \quad \rightarrow \quad \frac{\partial f}{\partial x} = 1, \quad \frac{\partial f}{\partial y} = \pm 1$$

Therefore

$$\sigma_f = \sqrt{\sigma_x^2 + \sigma_y^2},$$

the absolute errors add.

Example 2

$$f(x, y) = \frac{x}{y} \quad \rightarrow \quad \frac{\partial f}{\partial x} = \frac{1}{y} = \frac{f}{x}, \quad \frac{\partial f}{\partial y} = -\frac{x}{y^2} = -\frac{f}{y}$$

Therefore

$$\frac{\sigma_f}{f} = \sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2},$$

the relative errors add.

References

- [1] P. Horowitz and W. Hill, *The Art of Electronics*, Cambridge University Press (New York, 1980), pp. 193-194.
- [2] R. E. Simpson, *Introductory Electronics for Scientists and Engineers*, 2nd Ed., Allyn and Bacon, Inc. (Boston, 1987)