1. Consider a triangle wave of period T and amplitude A (peak to peak amplitude: $2A$).

$$f(t) = \begin{cases} -2A - \frac{4A}{T}t & -\frac{T}{2} \leq t < -\frac{T}{4} \\ \frac{4A}{T}t & -\frac{T}{4} \leq t < \frac{T}{4} \\ 2A - \frac{4A}{T}t & \frac{T}{4} \leq t < \frac{T}{2} \end{cases} \tag{1}$$

Develop the Fourier Series of the triangle wave.

Solution:

We introduce $x = \omega \cdot t = 2\pi/T \cdot t$, which implies that the relevant break points of the function are at $x_1 = \omega \cdot (T/4) = \pi/2$ and $x_2 = \omega \cdot (T/2) = \pi$.

$$f(x) = \begin{cases} -2A\left[1 + \frac{x}{\pi}\right] & -\pi \leq x < -\frac{\pi}{2} \\ 2A\frac{x}{\pi} & -\frac{\pi}{2} \leq x < \frac{\pi}{2} \\ 2A\left[1 - \frac{x}{\pi}\right] & \frac{\pi}{2} \leq x < \pi \end{cases} \tag{2}$$

The function is odd, $f(-x) = -f(x)$, so no constant terms or terms of the form $\cos(nx)$ [$= \cos(2\pi nt/T)$ in the original coordinates] may appear in the solution,

$$a_0 = a_n = 0. \tag{3}$$

It remains to determine the b_n coefficients. Again due to the symmetry we get equal contribution from the intervals $x = \omega t \in [-\pi, 0]$ and $x = \omega t \in [0, \pi]$.

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi/2}\left(2A\,\frac{x}{\pi}\right)\,\sin(nx)\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}\left(2A\left[1 - \frac{x}{\pi}\right]\right)\sin(nx)\,dx \\ &= \frac{4A}{\pi}\int_{\pi/2}^{\pi}\sin(nx)dx + \frac{4A}{\pi^2}\left[\int_0^{\pi/2}x\sin(nx)dx - \int_{\pi/2}^{\pi}x\sin(nx)dx\right] \\ &= \frac{4A}{\pi}\left[-\frac{\cos(nx)}{n}\right]_{\pi/2}^{\pi} + \frac{4A}{\pi^2}\left[\frac{\sin(nx)}{n^2} - \frac{x\cos(nx)}{n}\right]_0^{\pi/2} - \frac{4A}{\pi^2}\left[\frac{\sin(nx)}{n^2} - \frac{x\cos(nx)}{n}\right]_{\pi/2}^{\pi} \\ &= \frac{4A}{\pi}\left[-\frac{\cos(n\pi)}{n} + \frac{\cos(n\pi/2)}{n}\right] + \frac{8A}{\pi^2}\left[\frac{\sin(n\pi/2)}{n^2} - \frac{\pi\cos(n\pi/2)}{2n}\right] + \frac{4A}{\pi^2}\left[\frac{\pi\cos(n\pi)}{n}\right] \\ &= \frac{8A}{\pi^2 n^2}\sin\left(\frac{n\pi}{2}\right) \\ &= \frac{8A}{\pi^2 n^2}\begin{cases} (-1)^{(n-1)/2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \end{aligned} \tag{4}$$

Therefore, the Fourier Series of a Triangle wave is

$$f(t) = \frac{8A}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{(-1)^{(n-1)/2}}{n^2}\sin\left(\frac{2\pi n}{T}t\right) \tag{5}$$