FOURIER ANALYSIS

Lucas Illing

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Contents

1 Fourier Series

1.1 General Introduction

Consider a function $f(\tau)$ that is periodic with period T.

$$f(\tau + T) = f(\tau) \tag{1}$$

We may always rescale τ to make the function 2π periodic. To do so, define a new independent variable $t = \frac{2\pi}{T}\tau$, so that

$$f(t + 2\pi) = f(t) \tag{2}$$

So let us consider the set of all sufficiently nice functions $f(t)$ of a real variable t that are periodic, with period 2π. Since the function is periodic we only need to consider its behavior on one interval of length 2π, e.g. on the interval $(-\pi, \pi)$.

The idea is to decompose any such function $f(t)$ into an infinite sum, or series, of simpler functions. Following Joseph Fourier (1768-1830) consider the infinite sum of sine and cosine functions

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(nt) + b_n \sin(nt)] \tag{3}$$

where the constant coefficients a_n and b_n are called the Fourier coefficients of f. The first question one would like to answer is how to find those coefficients. To do so we utilize the **orthogonality** of sine and cosine functions:

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(nt)\, \cos(mt)\, dt &= \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos((m-n)t) + \cos((m+n)t)\right] dt \\
&= \begin{cases} 2\pi, & m = n = 0 \\ \pi, & m = n \neq 0 \\ 0, & m \neq n \end{cases} \\
&= \begin{cases} 2\pi, & m = n = 0 \\ \pi\delta_{mn}, & m \neq 0 \end{cases}
\end{aligned} \tag{4}$$

Similarly,

$$\int_{-\pi}^{\pi} \sin(nt)\ \sin(mt)\,dt = \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos((m-n)t) - \cos((m+n)t)\right] dt$$
$$= \begin{cases} 0 & m = 0 \\ \pi\delta_{mn} & m \neq 0 \end{cases} \tag{5}$$

and

$$\int_{-\pi}^{\pi} \sin(nt)\ \cos(mt)\,dt = \int_{-\pi}^{\pi} \frac{1}{2}\left[\sin((m-n)t) + \sin((m+n)t)\right] dt$$
$$= 0 \tag{6}$$

Using the orthogonality and the assumed expression for the infinite series given in Eq. (3), it follows that the Fourier coefficients are

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ \cos(nt)\,dt \tag{7}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ \sin(nt)\,dt \tag{8}$$

This initial insight by Fourier was followed by centuries of a work on the second obvious question: Are the RHS and LHS in Eq. (3) actually the same? Clearly one needs to determine for which class of functions f the infinite series on the right hand side of Eq. (3) will converge. That is, what is a sufficiently nice function f? The precise answer is not of concern here, it suffices to know that the Fourier series exists and converges for periodic functions of the type you are used to, e.g. functions for which first and second order derivatives exists almost everywhere, that are finite and have at most a finite number of discontinuities and zero crossings in the interval $(-\pi, \pi)$.

When determining a the Fourier series of a periodic function $f(t)$ with period T, any interval $(t_0, t_0 + T)$ can be used, with the choice being one of convenience or personal preference. For example, in the rescaled time coordinates considering the interval $(0, 2\pi)$ works just as well as considering $(-\pi, \pi)$ as we have done.

If a function is even so that $f(t) = f(-t)$, then $f(t)\sin(nt)$ is odd. (This follows since $\sin(nt)$ is odd and an even function times an odd function is an odd function.) Therefore, $b_n = 0$ for all n. Similarly, if a function is odd so that $f(t) = -f(-t)$, then $f(t)\cos(nt)$ is odd. (This follows since $\cos(nt)$ is even and an even function times an odd function is an odd function.) Therefore, $a_n = 0$ for all n.

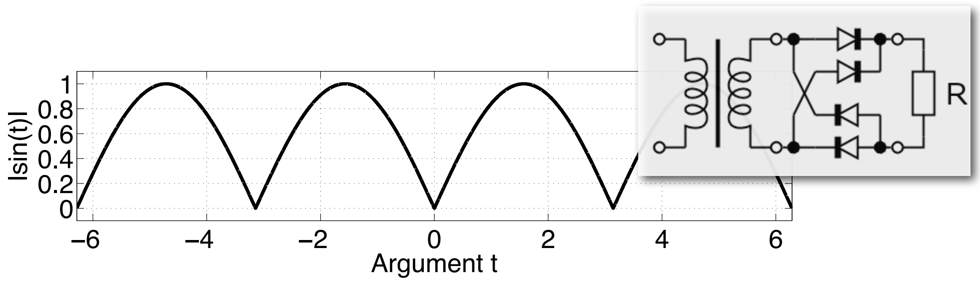

Figure 1: A full-wave-rectifier converts a sinusoidal input, $\sin(\omega t)$*, to* $|\sin(\omega t)|$.

Example - Rectified sine wave: A first step in converting AC-power from the power-grid to the DC-power that most devices need is to utilize a full-wave rectifier, such as the diode bridge shown in Fig. 1, which converts a sinusoidal input to an output that is the absolute value of the input sine-wave.

One notes immediately that for a sinusoidal input, the output of the rectifier is periodic with half of the period of the input. The fundamental frequency of the output is twice the input frequency. How can that be? The reason is that the circuit is *not* a linear circuit. The presence of diodes makes this circuit nonlinear and allows the circuit to shift power from the fundamental frequency to twice its value. One might wonder whether that is all that is happening. Does the output have contributions (power) at other frequencies? To answer this we look at the Fourier series of the output.

Since the output $f = |\sin(\omega t)|$ is even, *i.e.* $f(t) = f(-t)$, no terms of the form $\sin(n\omega t)$ will appear in the answer. It suffices to determine the a_n coefficients. For a_0 one obtains

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{0} -\sin(\omega t)\, d(\omega t) + \frac{1}{\pi}\int_{0}^{\pi} \sin(\omega t)\, d(\omega t) \\
&= \frac{2}{\pi}\int_{0}^{\pi} \sin(\omega t)\, d(\omega t) = \frac{4}{\pi} \qquad (9)
\end{aligned}
$$

and for the remaining a_n one gets

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} \sin(\omega t)\cos(n\omega t)\, d(\omega t) \\ &= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left[-\sin((n-1)\omega t) + \sin((n+1)\omega t)\right] d(\omega t) \\ &= \frac{1}{\pi}\left[\frac{1}{n-1}\left\{\cos(n\pi-\pi)-1\right\} + \frac{-1}{n+1}\left\{\cos(n\pi+\pi)-1\right\}\right] \\ &= \begin{cases} -\frac{4}{\pi}\frac{1}{n^2-1}, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases} \end{aligned} \tag{10}$$

Note, that the sine and cosine functions are orthogonal on the interval $(-\pi, \pi)$. They are not orthogonal on the interval $(0, \pi)$ and we do get a nonzero contribution for even n. To summarize the result,

$$|\sin(\omega t)| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2,4,6,\ldots}^{\infty} \frac{\cos(n\omega t)}{n^2-1}. \tag{11}$$

For an input with frequency f_0, the output has a DC-offset, the part that we really care about when building a DC-voltage supply. It has *no contribution* at $f = f_0$. It does have contributions at frequencies $2f_0, 4f_0, 6f_0, \ldots$.

1.2 Discontinuous Functions

In the above example, Eq. (11), the n^{th} coefficient decreases as $1/n^2$. This decay of the coefficients is in contrast to the Fourier series of a square wave

$$f_{sw}(t) = \frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n}\sin(n\omega t) \tag{12}$$

where the n^{th} coefficient falls off as $1/n$. This is true in general [1]

1. If $f(t)$ has discontinuities, the n^{th} coefficient decreases as $1/n$. The convergences is slow and many terms need to be kept to approximate such a function well.

[1] G. Raisbeck, Order of magnitude of Fourier coefficients. *Am. Math. Mon.* **62**, 149-155 (1955).

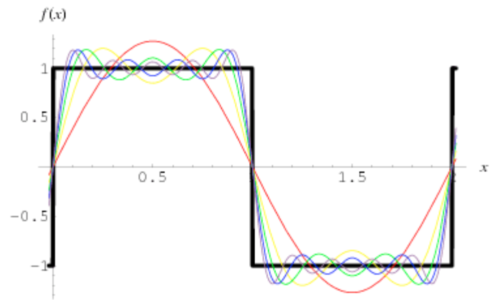

Figure 2: The Gibbs phenomenon is an overshoot (or "ringing") of Fourier series and other eigenfunction series occurring at simple discontinuities. Shown in color are the first few partial sums of the square-wave Fourier series. (Source mathworld.wolfram.com/GibbsPhenomenon.html)

This means that a function generator that generates square waves through the addition of sinusoidal waveforms needs to have a bandwidth (max. freq. it can generate) that is large compared to the frequency of the square-wave that is generated.

2. If $f(t)$ is continuous (although possibly with discontinuous derivatives) the n^{th} coefficient decreases as $1/n^2$.

There is another consequence of a discontinuity in $f(t)$ that can cause trouble in practical applications, where one necessarily only adds a finite number of sinusoidal terms. The nth partial sum of the Fourier series of a piecewise continuously differentiable periodic function f behaves at a jump discontinuity in a peculiar manner. It has large oscillations near the jump, which might increase the maximum of the partial sum above that of the function itself. It turns out that the Fourier series exceeds the height of a square wave by about 9 percent. This is the so-called **Gibbs phenomenon**, shown in Fig. 2. Increasing the number of terms in the partial sum does *not* decrease the magnitude of the overshoot but moves the overshoot extremum point closer and closer to the jump discontinuity.

You will have the opportunity to explore this in the lab.

1.3 Complex Fourier Series

At this stage in your physics career you are all well acquainted with complex numbers and functions. Let us then generalize the Fourier series to complex functions. To motivate this, return to the Fourier series, Eq. (3):

$$\begin{aligned} f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nt) + b_n \sin(nt)] \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \frac{e^{int} + e^{-int}}{2} + b_n \frac{e^{int} - e^{-int}}{2i} \right] \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - ib_n}{2} e^{int} + \sum_{m=-1}^{-\infty} \frac{a_{-m} + ib_{-m}}{2} e^{imt} \end{aligned} \tag{13}$$

where we substituted $m = -n$ in the last term on the last line. Equation (13) clearly suggests the much simpler complex form of the Fourier series

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n \, e^{in(2\pi f_0)t}. \tag{14}$$

with the coefficients given by

$$X_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \, e^{-in(2\pi f_0)t} \, dt \tag{15}$$

Here, the Fourier series is written for a complex periodic function $x(t)$ with arbitrary period $T = 1/f_0$. Note that the Fourier coefficients X_n are complex valued. It is seen from Eq. (13) that for a real-valued function $x(t)$ in Eq. (14) the following holds for the complex coefficients X_n

$$X_n = X_{-n}^* \tag{16}$$

where * denotes the complex conjugate.

This is all well, but you may wonder what to do about a function such as $e^{-\alpha t} \sin(\omega t)$. This function is not periodic, therefore not amenable to Fourier series analysis, but it is clearly oscillatory and very well behaved for $t > 0$ ($\alpha > 0$).

2 Fourier Transform

2.1 Definition

The Fourier transform allows us to deal with non-periodic functions. It can be derived in a rigorous fashion but here we will follow the time-honored approach of considering non-periodic functions as functions with a "period" $T \to \infty$. Starting with the complex Fourier series, *i.e.* Eq. (14) and replacing X_n by its definition, *i.e.* Eq. (15), we obtain

$$x(t) = \sum_{n=-\infty}^{+\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(\xi)\, e^{i\,2\pi\, n f_0\, (t-\xi)}\, d\xi \tag{17}$$

In a Fourier series the Fourier amplitudes are associated with sinusoidal oscillations at discrete frequencies. These frequencies are zero, for the DC term, the fundamental frequency $f_0 = 1/T$, and the higher harmonics $f = 2/T, 3/T, \ldots$. It is not hard to see that, when taking the limit $T \to \infty$, the spacing between adjacent frequencies will shrink to zero

$$(n+1)f_0 - n f_0 = f_0 = \frac{1}{T} \Rightarrow df \tag{18}$$

resulting in "Fourier amplitudes" at continuous frequencies $f = (0, \infty)$. Since for a given small frequency interval the number n of the discrete frequency harmonic that falls in this interval increases to infinity in the limit where the spacing between adjacent frequencies, T^{-1}, shrinks to zero, the product

$$f = \frac{n}{T} \tag{19}$$

is constant and will serve as the new continuous variable.

The above strongly suggest that in the limit the following should hold[2]

$$x(t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{\infty} x(\tau)\, e^{i2\pi f (t-\tau)}\, d\tau\, df. \tag{20}$$

[2] For a rigorous derivation see for example I.N.Sneddon, *Fourier Transform*, § 3.2 or Courant and Hilbert, *Methods of Mathematical Physics*, vol. 1, § 6.1

Equation (21) is called the **Fourier integral**. Note that the function x appears both on the LHS side and inside the double integral. We may rewrite the Fourier integral in several ways. For example, writing

$$x(t) = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{\infty} x(\tau)\, e^{-i2\pi f\tau}\, d\tau \right] e^{i2\pi ft} df. \tag{21}$$

suggests the introduction of the Fourier transform pair $x(t)$ and $\hat{x}(f)$:

$$\textbf{Fourier Transform:} \quad \hat{x}(f) = \mathcal{F}\left(x(t)\right) = \int_{-\infty}^{+\infty} x(t)\, e^{-i2\pi ft}\, dt, \tag{22}$$

$$\textbf{Inverse FT} \qquad x(t) = \mathcal{F}^{-1}\left(\hat{x}(f)\right) = \int_{-\infty}^{+\infty} \hat{x}(f)\, e^{i2\pi ft}\, df. \tag{23}$$

The function $\hat{x}(f)$ is the equivalent of the Fourier coefficients in the Fourier series. It is a function in the continuous *frequency domain* where $f \in (-\infty, +\infty)$. In a sense you may think of $x(t)$ as being made up of a continuum of sine waves. For obvious reasons, we talk about $x(t)$ as being in the *time domain* and the Fourier transform of $x(t)$, namely $\hat{x}(f)$, as being in the *frequency domain*. $x(t)$ and $\hat{x}(f)$ are two equivalent representations of the function of interest and these representations are connected by the Fourier transform. Note, that in general $\hat{x}(f)$ is a complex-valued function containing both *magnitude* and *phase* information.

Properties: Here are some useful properties of the Fourier transform that are relatively easily shown.

- As before, for a real-valued function $x(t)$, it holds that

$$\hat{x}(f) = \hat{x}(-f)^*. \tag{24}$$

- If function $x(t)$ is real, then the Fourier transform of $x(-t)$ is

$$\mathcal{F}[x(-t)] = \hat{x}(-f) = \hat{x}(f)^*. \tag{25}$$

- If function $x(t)$ is real and either even or odd, then the Fourier transform of $x(t)$ is also either even or odd.

- Using the definition of the Fourier Transform and integration by parts, it may be shown that

$$\mathcal{F}\left[\frac{d^n x(t)}{dt^n}\right] = (-i2\pi f)^n \hat{x}(f). \tag{26}$$

- The **Parseval** relation guarantees that the total power of a function is the same in the time- and frequency-domain, because the norms of the pair of Fourier transforms in time and frequency domains are equal, that is
$$||x||^2 = ||\hat{x}||^2 \tag{27}$$
with
$$||x||^2 = \int_{-\infty}^{+\infty} |x(t)|^2 dt \qquad ||\hat{x}||^2 = \int_{-\infty}^{+\infty} |\hat{x}(f)|^2 df. \tag{28}$$

Cos & Sin: It turns out that Fourier transform pairs are well defined not only for nice functions, such as square integrable functions, but also for distributions such as the δ-function. A definition of the delta function in terms of an integral is suggested by the Fourier integral, Eq. (21), if we simply change the order of integration to

$$x(t) = \int_{-\infty}^{+\infty} x(\xi) \underbrace{\int_{-\infty}^{\infty} e^{i2\pi f(t-\xi)}\, df}_{\delta(t-\xi)}\, d\xi \tag{29}$$

Then it must also be true that

$$\mathcal{F}\left(e^{i2\pi f_0 t}\right) = \int_{-\infty}^{+\infty} e^{-i2\pi(f-f_0)t}\, dt = \delta(f - f_0). \tag{30}$$

Therefore,

$$\begin{aligned} \hat{x}_{\cos} &= \mathcal{F}\left(\cos(2\pi f_0 t)\right) \\ &= \int_{-\infty}^{+\infty} \frac{e^{i2\pi f_0 t} + e^{-i2\pi f_0 t}}{2}\, e^{-i2\pi f t}\, dt \\ &= \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right] \end{aligned} \tag{31}$$

and

$$\begin{aligned} \hat{x}_{\sin} &= \mathcal{F}\left(\sin(2\pi f_0 t)\right) \\ &= \frac{i}{2}\left[-\delta(f - f_0) + \delta(f + f_0)\right] \end{aligned} \tag{32}$$

Both sin and cos result in a Fourier transform that is exactly zero except at $f = f_0$ and $f = -f_0$. The distinguishing feature of sin and cos is their phase and this results in the different coefficients in front of $\delta(f - f_0)$ and $\delta(f + f_0)$. This example also shows that the Fourier transform is generally complex.

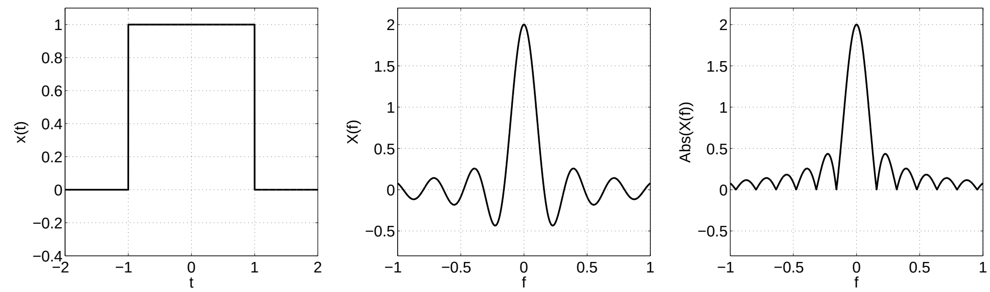

Figure 3: Box function and its Fourier transform

Box Function: Consider the Fourier transform of a box-function

$$b_T(t) = \begin{cases} 1 & t \in [-T/2, T/2] \\ 0 & \text{otherwise} \end{cases} \tag{33}$$

$$\begin{aligned} \hat{b}_T(f) &= \int_{-T/2}^{+T/2} e^{-i2\pi f t}\, dt \\ &= \frac{e^{-i\pi f T} - e^{i\pi f T}}{-2\pi i f} \\ &= T\,\frac{\sin(\pi f T)}{\pi f T} = T\,\mathrm{sinc}(\pi f T) \end{aligned} \tag{34}$$

The result is shown in Fig. 3. In physical optics the diffraction pattern amplitude is described by the Fourier transform of the diffracting element. A slit is described by the box function $b_a(x)$ and therefore the diffraction pattern by $\hat{b}_T(k)$.

2.2 The issue of convention

In signal processing the Fourier Transform pair is usually defined as above in terms of ordinary frequency (Hertz) and the "- i" in the forward Fourier transform.

$$\textbf{FT} \qquad \hat{x}_1(f) = \mathcal{F}\left[x(t)\right] \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} x(t)\, e^{-i\,2\pi f\,t}\, dt \tag{35}$$

$$\textbf{IFT} \qquad x(t) = \mathcal{F}^{-1}\left[\hat{x}(f)\right] = \int_{-\infty}^{\infty} \hat{x}_1(f)\, e^{i\,2\pi f\,t}\, df \tag{36}$$

Now the above is just a convention and many scientists prefer to use the angular frequency ω (rad/s) as integration variable. There are again several possibilities. For example in modern physics one often finds the following symmetric definition[3]

$$\textbf{FT} \qquad \hat{x}_2(\omega) = \mathcal{F}\left[x(t)\right] \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) e^{i\omega t}\, dt = \frac{1}{\sqrt{2\pi}}\, \hat{x}_1\left(\frac{-\omega}{2\pi}\right) \tag{37}$$

$$\textbf{IFT} \quad x(t) = \mathcal{F}^{-1}\left[\hat{x}_2(\omega)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{x}_2(\omega) e^{-i\omega t}\, d\omega \tag{38}$$

Note that here the forward Fourier transform is defined with positive "i". This latter convention, the modern physics one, is the default in Mathematica.

In addition to these two, there are several other such widely used conventions. Now this is an unfortunate state of affairs but that is how it stands. As a result you always have to state what you mean when talking about the Fourier transform of a function $x(t)$. We will be using the signal-processing pair of equations, namely Eq. (35) and Eq. (36).

2.3 Convolution Theorem

The convolution theorem states that under suitable conditions the Fourier transform of a convolution is the pointwise product of Fourier transforms. And, the Fourier transform of the pointwise product of two functions is the convolution of the Fourier transform of each one of the two functions. In other words, convolution in one domain (e.g., frequency domain) equals point-wise multiplication in the other domain (e.g., time domain). This allows one to break apart problems into manageable pieces and is extremely useful. We will utilize it to explain spectral leakage. But let's start by considering the Fourier transform of the point-wise product of two functions in the time domain.

$$\mathcal{F}\left[x(t) \cdot y(t)\right] = \int_{-\infty}^{+\infty} x(t) \cdot y(t)\; e^{-i2\pi f t}\, dt$$

We now replace x and y by the corresponding inverse Fourier transform [Eq. (23)], change the order of integration, and use the definition of the delta

[3] e.g. see Arfken and Weber's *Mathematical Methods for Physicists*, Fifth Ed., §15.2, pg. 690

function:

$$\begin{aligned}
\mathcal{F}\left[x(t)\cdot y(t)\right] &= \int_{-\infty}^{+\infty} x(t)\cdot y(t)\; e^{-i2\pi f t}\, dt \\
&= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \hat{x}(\rho)\, e^{i2\pi\rho t}\, d\rho \cdot \int_{-\infty}^{+\infty} \hat{y}(\nu)\, e^{i2\pi\nu t}\, d\nu\right]\; e^{-i2\pi f t}\, dt \\
&= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \hat{x}(\rho) \left(\int_{-\infty}^{+\infty} \hat{y}(\nu) e^{i2\pi(\rho+\nu)t}\, d\nu\right)\, d\rho\right]\; e^{-i2\pi f t}\, dt \\
&= \int_{-\infty}^{+\infty} \hat{x}(\rho) \left(\int_{-\infty}^{+\infty} \hat{y}(\nu) \int_{-\infty}^{+\infty} e^{i2\pi(\rho+\nu-f)t}\, dt\, d\nu\right)\, d\rho \\
&= \int_{-\infty}^{+\infty} \hat{x}(\rho) \left(\int_{-\infty}^{+\infty} \hat{y}(\nu)\, \delta(\rho+\nu-f)\, d\nu\right)\, d\rho \\
&= \int_{-\infty}^{+\infty} \hat{x}(\rho)\; \hat{y}(f-\rho)\, d\rho.
\end{aligned} \tag{39}$$

The last line is the convolution of the Fourier transform $\hat{x}$ of $x(t)$ and the Fourier transform $\hat{y}$ of $y(t)$. The convolution is sometimes denoted by $*$,

$$\hat{x}(f) * \hat{y}(f) = \int_{-\infty}^{+\infty} \hat{x}(\rho)\; \hat{y}(f-\rho)\, d\rho$$

So, to write the result neatly

$$\mathcal{F}\left[x(t)\cdot y(t)\right] = \mathcal{F}\left[x(t)\right] * \mathcal{F}\left[y(t)\right] = \int_{-\infty}^{+\infty} \hat{x}(\rho)\; \hat{y}(f-\rho)\, d\rho. \tag{40}$$

The Fourier transform of the pointwise product of two functions is the convolution of the Fourier transform of each one of the two functions.

With that out of the way, we can discuss spectral leakage in a straightforward manner.

2.4 Spectral Leakage

Real life limitations have consequences:
1) A finite measurement time results in spectral leakage

We want to know the Fourier transform of an experimentally measured voltage $x(t)$ that can be described by $\cos(2\pi f_0 t)$, for example. In other words we are given a function in the time domain. We can then use Eq. (22) to calculate the corresponding function in the frequency domain, except that Eq. (22) requires us to know $x(t)$ for $t \in (-\infty, \infty)$. Even the most patient student will not measure an output forever, so one has to assume that $x(t) = \cos(2\pi f_0 t)$ is only given on a finite interval $t \in (-T/2, T/2)$.

So in reality, on a computer for example, we would evaluate the Fourier transform of this finite time series. We can discuss the consequences of the finite length of our time series by considering the Fourier transform of the product of $x(t)$ [defined for $t \in (-\infty, \infty)$] with the box-function $b_T(t)$, as shown in Fig. 4. Since we know the Fourier transform of the cosine-function and the box-function, respectively Eq. (31) and Eq. (34), we can utilize the convolution theorem to obtain the Fourier transform with little additional work.

$$
\begin{aligned}
\mathcal{F}\left[\cos(2\pi f_0 t) \cdot b_T(t)\right] &= \int_{-\infty}^{+\infty} \frac{1}{2}\left[\delta(\rho - f_0) + \delta(\rho + f_0)\right] \cdot T \,\mathrm{sinc}(\pi T(f - \rho))\, d\rho \\
&= \frac{T}{2}\left[\mathrm{sinc}(\pi T(f - f_0)) + \mathrm{sinc}(\pi T(f + f_0)\right] \qquad (41)
\end{aligned}
$$

It is seen from Eq. (41) and Fig. 4 that the the finite observation time results in a distribution of spectral power according to the sinc-function that is centered at the location of the delta-function that would have resulted from a Fourier-transform of an infinitely long time-series of $\cos(2\pi f_0 t)$.

The process of multiplying $\cos(2\pi f_0 t)$ by the box-function is referred to as windowing with a rectangle. Windowing of simple functions such as the cosine-function causes the Fourier transform to have non-zero values at frequencies other than f_0. This effect, commonly called spectral leakage is worst near f_0 and least at frequencies farthest away from f_0.

If there are two sinusoidal functions with different frequencies leakage interferes with ones ability to distinguish the two functions in the spectrum and can be sufficient to make the two sinusoids unresolvable. The situation is most challenging if the frequencies of the two sine-functions are close and if they have very different amplitudes. Distinguishing two sinusoids with nearly identical frequencies and equal amplitude requires *high resolution* and distinguishing two sinusoids with vastly different signal amplitudes requires a large *dynamic range*.

The rectangle window (box-function) has excellent resolution characteristics for signals of comparable strength, but it is a poor choice for signals of disparate amplitudes. This characteristic is sometimes described as high resolution and low-dynamic-range.

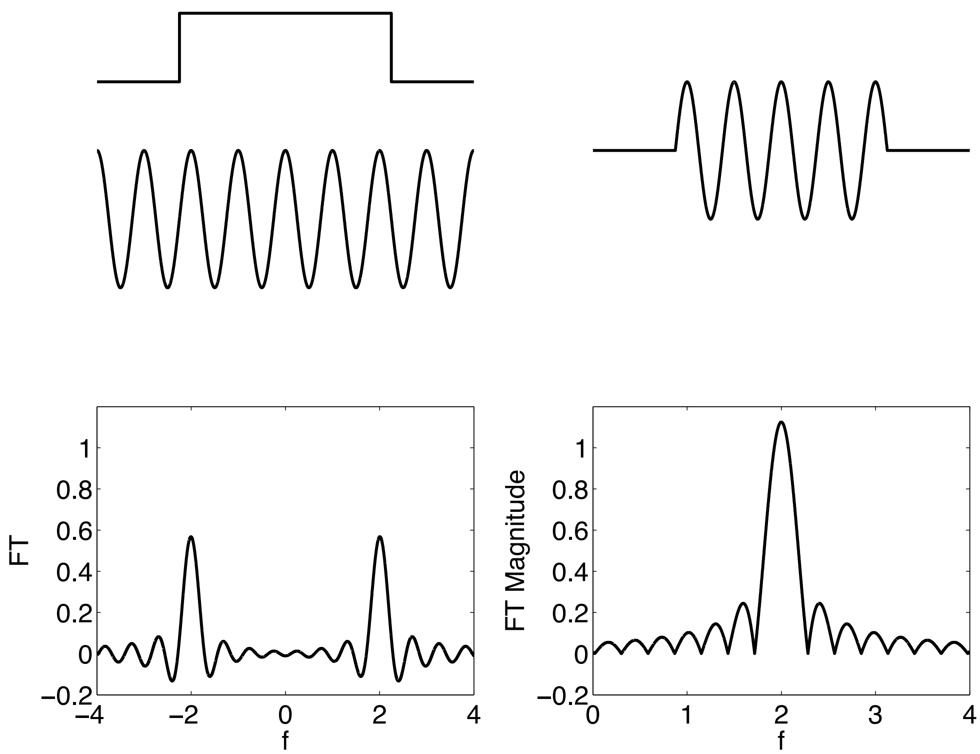

Figure 4: (Top) The product of a cosine-function and a box function. (Bottom) The Fourier transform of the product

The choice of a box-function-window was motivated by the fact that it simply corresponds to a finite time series. We are however free to multiply the time series that we obtain from measurements by a window function of our choice. There are many window functions. The decision about which is he best window to use will depend on the particular application and the desired trade-off between dynamic-range and resolution. For example, the flat top window (Fig. 5a) is a high-dynamic-range low-resolution window, at the other extreme from the rectangle-window, and the Hann window (Fig. 5b) is somewhere in the middle, providing both decent dynamic-range and resolution. You will have opportunities to explore the effect of windowing in lab.

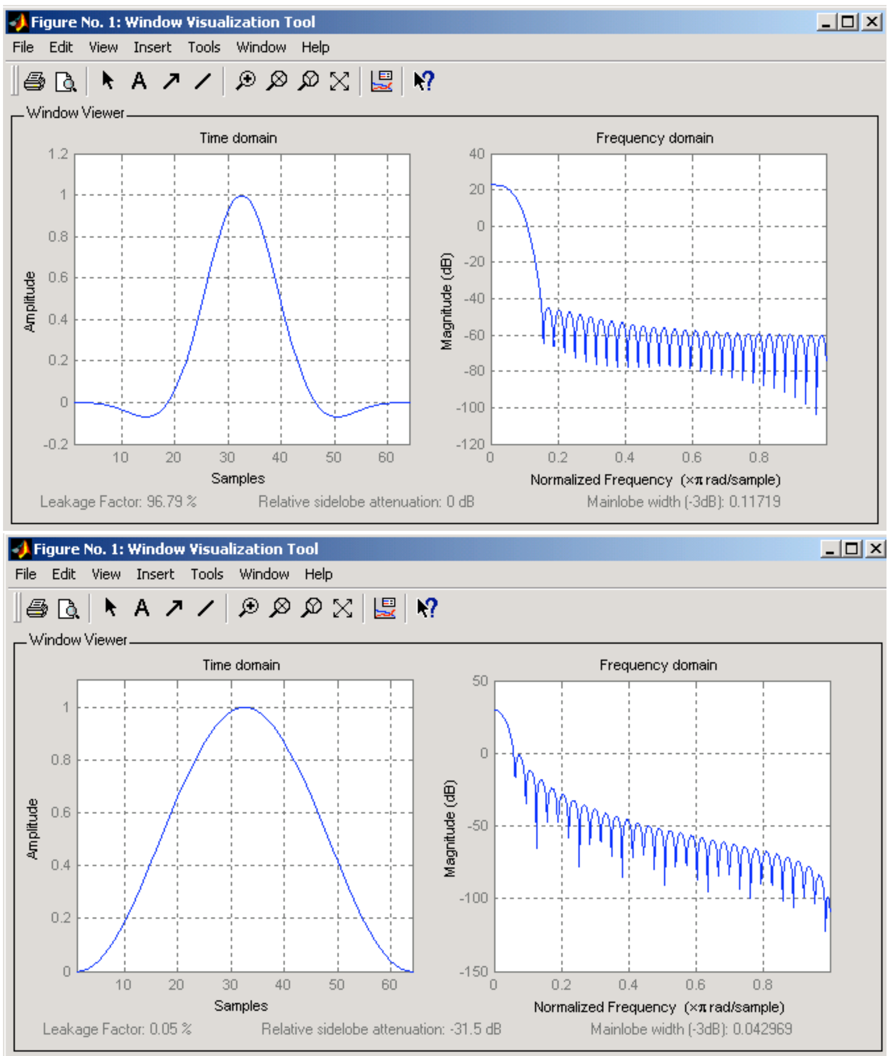

Figure 5: (Top) Flat-top window and its Fourier transform (Bottom) Hann window and its Fourier transform

3 Discrete Time

Real life limitations have consequences:
2) Finite-time data is taken at discrete times. The spectrum is then periodic and typically discrete.

3.1 Discrete Time Fourier Transform

Since data is typically taken at integer times $t_j = t_0, t_1, \ldots, t_{N-1}$ one needs to consider a discrete version of the Fourier transform. The easiest to treat case and the only one that we will consider is the case of N equal time samples, *i.e.*

$$t_j = j\Delta t \quad j = 0, 1, 2, \ldots, N-1 \tag{42}$$

Here Δt is the time between two neighboring data points in the time series and $f_s = 1/\Delta t$ is the sampling frequency, the number of samples taken per second. The discrete time Fourier-transform (DTFT) for a finite-length discrete-time data set

$$X(0), X(\Delta t), X(2\Delta t), \ldots, X((N-1)\Delta t) \tag{43}$$

is then simply an approximation to the continuous Fourier transform [Eq. (4)] and is given by

$$\textbf{DTFT} \quad \hat{X}(f) = \sum_{j=0}^{N-1} X(t_j)\, e^{-i\, 2\pi f\, t_j}. \tag{44}$$

One way to obtain this result is to consider the DTFT as arising from a Fourier transform of the product of $x(t)$ with the *Dirac comb*. The Dirac comb, often called the sampling function or impulse train, is in general

$$\Delta_{\Delta t}(t) \stackrel{\text{def}}{=} \sum_{j=-\infty}^{\infty} \delta(t - j\Delta t) \tag{45}$$

and for the finite time series $\{t_j\}$ that we are considering

$$\Delta_j(t) \stackrel{\text{def}}{=} \sum_{j=0}^{N-1} \delta(t - t_j) \tag{46}$$

The Fourier transform of the product is then easily evaluated

$$\begin{aligned}\mathcal{F}\left[X(t_j)\right] &= \mathcal{F}\left[x(t) \cdot \Delta_j(t)\right] \\ &= \int_{-\infty}^{+\infty} x(t) \cdot \sum_{j=0}^{N-1} \delta(t - t_j)\; e^{-i2\pi f t}\, dt \\ &= \sum_{j=0}^{N-1} X(t_j)\; e^{-i2\pi f t_j} \end{aligned} \tag{47}$$

and yields Eq. (44).

Periodicity: Sampling $x(t)$ causes its DTFT-spectrum to become periodic with frequencies repeating every f_s, where f_s is the sample rate. This follows directly from the periodicity of

$$e^{-i2\pi f t_j} = e^{-i2\pi f j \Delta t} = e^{-i2\pi f j / f_s}. \tag{48}$$

That is,

$$e^{-i2\pi(f + k f_s) t_j} = e^{-i(2\pi f t_j + 2\pi k f_s\, j / f_s)} = e^{-i2\pi f t_j} e^{-i2\pi k j} = e^{-i2\pi f t_j} \tag{49}$$

where both j and k are arbitrary integers. Therefore:

$$\hat{X}(f + k f_s) = \hat{X}(f) \tag{50}$$

For the DTFT f is continuous and one could evaluate $\hat{X}(f)$ at any desired frequency. A priory f can be anything, but since $\hat{X}(f)$ is periodic one only needs to consider a frequency interval of length f_s. Typically we are interested in the region $[-f_s/2, f_s/2]$, because $f_s/2$ is the Nyquist frequency, the maximum frequency of the original signal $x(t)$ that can be resolved when the signal is sampled with sampling rate f_s. This follows from the fact that one needs at least two samples per cycle, a peak and a trough, in order to estimate a frequency.

$$f_{Nyquist} = \frac{f_s}{2}. \tag{51}$$

To summarize, the Discrete-Time-Fourier-transform estimates the spectrum at continuous frequencies based on a discrete-time data set. The spectrum, *i.e.* the signal in the Fourier domain, is periodic. The DTFT does therefore the reverse of a Fourier-series, which produces Fourier-amplitudes at discrete frequencies corresponding to a *periodic* continuous time signal. Now if our signal x is *periodic and discrete*-time, then the spectrum will also be discrete, and the corresponding transform is called the *Discrete-Fourier transform (DFT).*

3.2 Discrete Fourier Transform (and FFT)

So, if the signal is periodic, $x(t+N\Delta t) = x(t)$, then, just as was the case with the Fourier series, the signal contains power only at its fundamental frequency $1/(N\Delta t)$ and higher harmonics. In other words, for a periodic signal with period $T = N\Delta t$ measured at N discretely sampled times the corresponding frequencies are also discretized with the spacing between adjacent frequencies being given by $\Delta f = 1/(N\Delta t) = f_s/N$.

Due to the periodicity of f, we can choose any interval of length $f_s = N\Delta f$. Typically one would like to chose (N even)

$$f_k = k\,\Delta f \quad k = -\frac{N}{2}+1,\ldots,0,\ldots,\frac{N}{2} \tag{52}$$

where the frequency region is bracketed by the Nyquist frequency and its negative, *i.e.* $[-f_s/2, f_s/2]$. We do not consider $f = -\frac{N}{2}\Delta f$ because it is the same as $f = \frac{N}{2}\Delta f$ due to periodicity ($-\frac{N}{2}\Delta f = -\frac{N}{2}\Delta f + f_s = \frac{N}{2}\Delta f$).

However, since computers like to work with positive indices, one uses the periodicity to map the interval $-\frac{N}{2}+1$ to -1 onto the interval $\frac{N}{2}+1$ to N-1 to obtain

$$f_k = k\,\Delta f \quad k = 0,1,\ldots,N-1. \tag{53}$$

As shown in J. Essick's book (chapter 10 pg. 3-5), the DFT pair is then given by

$$\textbf{DFT:} \quad \hat{X}(f_k) = \sum_{j=0}^{N-1} X(t_j)\, e^{-i\,2\pi f_k\, t_j} \quad k = 0,\ldots,N-1 \tag{54}$$

$$\textbf{IDFT:} \quad X(t_j) = \sum_{k=0}^{N-1} \hat{X}(f_k)\, e^{i\,2\pi f_k\, t_j} \quad j = 0,\ldots,N-1 \tag{55}$$

The DFT can be efficiently performed using the **Fast Fourier Transform (FFT)**, which is the name for a particular computational algorithm developed to compute a DFT. The FFT requires the size of the data set to be a power of two, $N = 2^m$ with m an integer. The FFT reduces the required number of operations from the brute-force approach that requires order N^2 operations to $N\log_2 N$ such operations. This is a huge increase of speed for large data sets. For example, for a data set consisting of about one hundred Megasamples (2^{27}) the FFT is 5 Million times faster. Nowadays data sets with millions of sample points are common both in bench-top and numeric experiments.

Due to the immense computing advantage of the FFT, it forms the basis of many numerical algorithms. For example, it is used not only to estimate the spectrum of periodic data sets but any discretely sampled function.

For time series with a period commensurate with the observation time $N\Delta t$, the DFT (or FFT) is "exact" in the sense that the spectrum contains delta-peaks. This follows from the fact that for a function that is periodic on the interval given by the observation time, the Fourier amplitudes appear at discrete frequencies $f = n/(N\Delta t)$ and it is only at those frequencies that we sample the Fourier transform. Now, it is still true that the finite observation time implies that, instead of a delta peak, there will be, in principle, a sinc-function centered at each of these frequencies. However, each of these sinc functions will only be sampled at its central peak and at the the frequencies for which it is zero. This follows because the sinc functions are centered at $f = n/(N\Delta t)$ and have zeros at $f \pm k\Delta f$, with $\Delta f = k/(N\Delta t)$ (because the observation time is $(N\Delta t)$). Since Δf is our sampling stepsize in the frequency domain, these zeros of each sinc function coincide with the places where the frequency space is sampled. As a result, the Fourier-transform looks exactly like a series of delta peaks that one would expect for an infinite periodic signal.

For time series that are non-periodic or have a non-commensurate frequency, the observation time is not a period of the signal. In this case one observes spectral leakage when performing the FFT. The leakage is described by the Fourier-transform of the windowing function, but discretely sampled in frequency space. These samples will be non-zero. This is entirely equivalent to the leakage for the continuous Fourier transform that we discussed above. You will have the opportunity to explored spectral leakage in lab when going through J. Essick's book.

4 Executive Summary

The type of Fourier analysis that you should consider depends on your input x (see table below). For experiments (even numeric ones) you will essentially always use the Fast Fourier Transform (FFT). So take 2^n data points. Generally, there will be spectral leakage. You can reduce leakage artifacts by applying an appropriate window function to the data as part of the Fourier analysis.

Time Domain	Transform Type	Frequency Domain
$x(t)$ continuous, non-periodic	Fourier Transform	$\hat{x}(f)$ continuous, non-periodic
$x(t)$ continuous, periodic	Fourier Series	$\hat{x}(f_k)$ discrete, non-periodic
$x(t_j)$ discrete, non-periodic	Discrete Time Fourier Transform	$\hat{x}(f)$ continuous, periodic
$x(t_j)$ discrete, periodic	Discrete Fourier Transform (Fast Fourier Transform)	$\hat{x}(f_k)$ discrete, periodic