Problem 2.1

a. Let \( C(r) = r^n \), then:
\[
\frac{\partial}{\partial r} (r^n) = n r^{n-1} \quad \text{and} \quad \frac{\partial}{\partial r} (\frac{\partial}{\partial r} (r^n)) = \frac{\partial}{\partial r} (n r^{n-1}) = n(n-1) r^{n-2}.
\]
In order to solve (1):
\[
\frac{\partial}{\partial r} (r^n) = n r^{n-1} \quad \text{and} \quad \frac{\partial}{\partial r} (\frac{\partial}{\partial r} (r^n)) = n(n-1) r^{n-2}.
\]
we need \( n = 1 \), \( n = -2 \) \& two solutions are:
\[
C(r) = r \quad \text{and} \quad C(r) = \frac{1}{r^2}.
\]

b. The left sketch is continuous \& finite at \( r = \infty \).

c. The right sketch is continuous, derivative - continuous at \( R \), \& is bounded on the right \( (> R) \).

d. \( \hat{a}^+ = \hat{a}^T, \hat{a}^+ = \hat{a} \), then:
\[
[\hat{a}, \hat{a}^T] = (\hat{a} \hat{a}^T - \hat{a}^T \hat{a}) = \hat{a}^+ \hat{a}^+ - \hat{a}^T \hat{a}^T = \hat{a} \hat{a} - \hat{a}^T \hat{a} = \hat{a} \hat{a} - \hat{a}^T \hat{a} = -[\hat{a}, \hat{a}^T]
\]

Additionally, \([\hat{a}, \hat{a}^T] = -[\hat{a}, \hat{a}^T] \) means that \([\hat{a}, \hat{a}^T] \) is anti-Hermitian.

e. Take \( \hat{a} = x \rho \), then:
\[
\frac{\partial}{\partial x} \langle x | \rho \rangle = \frac{1}{i} \langle [H, x \rho] \rangle = \infty
\]
The commutator is:
\[
[H, x \rho] = \frac{\partial^2}{\partial x^2} x \rho + [V(x), x \rho]
\]

- We need:
\[
\frac{\partial}{\partial x} [V(x), x \rho] = [x \frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2} \rho, x \rho] + [V(x), x \rho]
\]

- \( \rho = \frac{\partial}{\partial x} x \rho - x \frac{\partial^2}{\partial x^2} \rho \)
\[
= \frac{\partial}{\partial x} x \rho - x \frac{\partial^2}{\partial x^2} \rho = \rho \frac{\partial}{\partial x} - 2 i \rho \frac{\partial^2}{\partial x^2}
\]
\[
= -2 i \rho \frac{\partial^2}{\partial x^2}
\]

and
\[
[V(x), x \rho] = V(x) x \frac{\partial}{\partial x} x \rho - x \frac{\partial^2}{\partial x^2} (V(x)) x \rho
\]

- \( V(x) x \frac{\partial}{\partial x} x \rho - x \frac{\partial^2}{\partial x^2} (V(x)) x \rho = i \frac{\partial}{\partial x} x \rho \cdot V(x)
\]
\[
= + i V(x) \frac{\partial}{\partial x} x \rho
\]
So
\[
[H, x \rho] = i \rho x V', \quad \text{then}
\[
[H, x \rho] = -2 i \rho \frac{\partial^2}{\partial x^2} + i \rho x V'
\]

\[\langle [H, x \rho] \rangle = -2 i \rho \langle T \rangle + i \rho \langle x V' \rangle = 0 \Rightarrow 2 \langle T \rangle = \langle x V' \rangle\]
Problem 2.2

We have: \(-\frac{k^2}{2m} \nabla^2 \psi(x) = E^2 \psi(x)\)

or
\[ \frac{1}{a^2} \frac{d}{d\phi} \frac{d}{d\phi} \psi = -\frac{2mE^2}{k^2} \frac{d^2}{d\phi^2} \psi \]

where \(k^2 = \frac{2mE^2}{\hbar^2} \), then

\[ \frac{d^2\psi}{d\phi^2} = -\frac{\hbar^2}{\kappa^2} \psi \]

\[ \psi(\phi) = A e^{ik\phi} + B e^{-ik\phi} \]

If we require that \(\psi(\phi+2\pi) = 2\psi(\phi)\), then

\[ A e^{i\kappa \phi} + B e^{-i\kappa \phi} = 2A e^{i\kappa \phi} \]

so we must have \(e^{i\kappa \phi} = e^{-i\kappa \phi} = 1 \Rightarrow \kappa a = n \) for integer \(n\).

Then:
\[ k^2 \psi^2 = \frac{n^2}{a^2} \]
\[ \frac{2mE^2}{\kappa^2} = n^2 \Rightarrow E_n = \frac{\sqrt{2mE^2}}{2ma^2} \]

\[ \Delta E = E_2 - E_1 = \frac{3\hbar^2}{2ma^2} = 2\pi \kappa \lambda \Rightarrow \lambda = \frac{3\hbar^2}{4\pi \kappa ma^2} \]

\[ \lambda = \frac{\lambda_n}{c} = \frac{4\pi m \kappa^2 c}{\hbar} \]

Problem 2.3

Schrödinger's eqn. reads: \(-\frac{1}{2m} \frac{\partial^2}{\partial x^2} \psi + V_0 \sin(\lambda x) \psi = i \hbar \frac{\partial}{\partial t} \psi \) for \(\psi(x,t)\).

and we have a natural separation of variables here:

\[ \psi(x,t) = \psi(x) \phi(t) \]

\[ \frac{1}{2m} \psi'' + E\psi = \frac{1}{2m} \phi'' + E\phi = 0 \]

the spatial solution is: \(\psi(x) = A \cos(kx) + B \sin(kx)\) or \(k \frac{2mE^2}{\hbar^2} \)

and we have boundary conditions: \(\psi(0) = 0\), \(\psi(\pi) = 0\) \(\Rightarrow \kappa a = n\pi\) for an integer \(n\). Then

\[ E_n = \frac{n^2 \hbar^2}{2ma^2} \]

For the temporal part:
\(i \hbar \frac{\partial}{\partial t} = V_0 \sin(\lambda x) \Rightarrow\)

\[ \frac{\partial}{\partial t} \log(\phi) = \frac{\partial}{\partial t} \left[ -\frac{V_0}{\omega} \cos(\omega t) + E t \right] \]

\[ \log(\phi) = -\frac{\omega t}{\omega} + \frac{V_0}{\omega} \cos(\omega t) = \phi(t) = \frac{\sqrt{a}}{\omega} \cos(\omega t) \]

\[ \psi(x,t) = \sqrt{\frac{a}{\omega}} \sin(\frac{\pi x}{a}) \exp\left( \frac{i}{\hbar} \left[ -\frac{\omega^2 x^2}{2ma^2} + \frac{V_0}{\omega} \cos(\omega t) \right] \right) \]