Problem 10.1

We still have: \(-\frac{d^2}{dx^2} 2\pi x = E \Phi(x)\)

but this time, the boundary conditions are:

\(\Phi(-\pi/2) = \Phi(\pi/2) = 0\).

So start w/ the general solution:

\[ \Phi(x) = Ae^{ikx} + Be^{-ikx} \quad w/ \quad k = \sqrt{\frac{E}{2\pi}} \]

**Impose the left boundary:** \(\Phi(-\pi/2) = Ae^{-ik\pi/2} + Be^{ik\pi/2} = 0\)

So \(B = -Ae^{ikx}\), \(A = 0\) then:

\[ \Phi(x) = A[e^{ikx} - e^{-ikx}] \]

The the right-hand boundary condition is:

\[ \Phi(\pi/2) = A[e^{ik\pi} - e^{-ik\pi}] \]

**For this to be zero, we need** \(e^{ik\pi/2} = e^{-ik\pi/2} \Rightarrow e^{ik\pi/2} = 1\)

This gives us \(k\pi = n \cdot 2\pi\) or \(k = \frac{2n}{\pi}\) for integer \(n\).

Now, we can reduce \(\Phi(x)\) to trigonometric functions:

\[ \Phi(x) = A[e^{ikx} - e^{-ikx}] = A[e^{ikx} - e^{-ikx}] = \frac{(-1)^n}{2} \]

For \(n\) odd, then, we have:

\[ \Phi_n(x) = A[e^{ik(n\pi/2)}x + e^{-ik(n\pi/2)}] = \cos\left(\frac{n\pi x}{a}\right) \]

while for \(n\) even,

\[ \Phi_n(x) = A[e^{ik(n\pi/2)}x - e^{-ik(n\pi/2)}] = \sin\left(\frac{n\pi x}{a}\right) \]

The normalization, in either case, is the usual \(\frac{\sqrt{2}}{a}\), so we have:

\[ \Phi_n(x) = \begin{cases} \frac{\sqrt{2}}{a} \cos\left(\frac{n\pi x}{a}\right) & \text{odd} \\ \frac{\sqrt{2}}{a} \sin\left(\frac{n\pi x}{a}\right) & \text{even} \end{cases} \]

In either case (\(n\) even or odd) we have the spectrum: \(k = \frac{n\pi}{2a} \Rightarrow \sqrt{\frac{E}{2\pi}} = \frac{n\pi}{2a}\),

\[ E_n = \frac{n^2 \pi^2 \hbar^2}{2ma} \]

(Note that you can also take the \(\pi/2\) well solution 6 shift: \(x \rightarrow x - \pi/2\)

\(i\) use trig identities to get these quickly.)
After the measurement of $E = \frac{n^2\hbar^2}{2m}$, we know the particle is in the $n=1$ state of the infinite square well. It remains in the state:

$$\psi(x) = \begin{cases} \frac{1}{\sqrt{3}} \cos \left( \frac{\pi x}{a} \right) & -\frac{a}{2} \leq x \leq \frac{a}{2} \\ 0 & \text{else} \end{cases} \quad (\text{x})$$

since this is a stationary state. When we turn on the harmonic oscillator potential, $(\text{x})$ represents our initial state. $\psi_0(x,0)$, so we know that a measurement of $E = \hbar \omega n$ corresponds to the ground state of the harmonic oscillator, and will be measured with probability given by $A_0^* A_0$ in the decomposition:

$$\psi(x,t) = \sum_{n=0}^{\infty} A_n \psi_n(x) e^{-\frac{\hbar^2 \omega}{2m} \frac{n^2}{a^2}}$$

where these coefficients are set by the requirement:

$$\psi(x,0) = \sum_{n=0}^{\infty} A_n \psi_n(x) = \begin{cases} \frac{1}{\sqrt{3}} \cos \left( \frac{\pi x}{a} \right) & -\frac{a}{2} \leq x \leq \frac{a}{2} \\ 0 & \text{else} \end{cases} = \tilde{\psi}(x)$$

so we need to compute: $A_0 = \int_{-\infty}^{\infty} \psi_0(x) \tilde{\psi}(x) \, dx$

then $P(\omega) = A_0^* A_0$. 

\[
\int_{-\frac{a}{2}}^{\frac{a}{2}} \left( \frac{m \omega}{\hbar} \right)^{\frac{1}{2}} e^{-\frac{m \omega x^2}{2\hbar}} \frac{1}{\sqrt{2}} \cos \left( \frac{\pi x}{a} \right) \, dx
\]