Varieties of Parthood

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1 Introduction

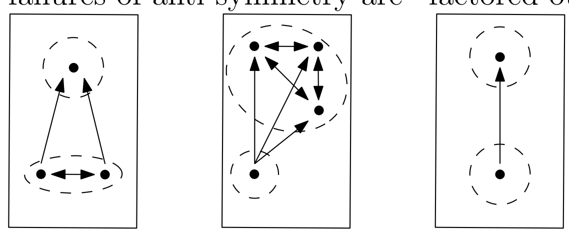
The basic intuitive idea driving the work in this paper is that there are multiple relations that are *like* the relation of part-to-whole, and various general principles that govern and organize these relations. A variation of this idea is that there are multiple ways of being a part; another is that there are multiple relations that are equally good candidates for being called "the" part-whole relation.

To illustrate one version of this idea, we might think that there are distinct primitive manners of being a part: for example, Socrates' nose is a part of Socrates in one manner (call it R), and Socrates is a part of the set $\{\text{Socrates}\}$ in a different manner (call it S). And it seems plausible that Socrates' nose is not a part of the set in any primitive manner. But it is a part in some derivative manner. We might say that there is a derivative form of parthood that is the "union" of R and S, call it $R \cup S$. Now the nose does not bear even this relation to the set. But it does bear the ancestral of this relation to the set. So we have a fourth, more inclusive or "tolerant" form of being a part.

There is a much more detailed discussion of informal versions of the ideas pursued here in [Hovda, 2016]. The basic outlook is deeply indebted to Kit Fine's [Fine, 2010], though there are some differences of approach. A key difference throughout is that we do not assume that all forms of parthood are anti-symmetric. As a brief motivation, we suggest that it is a reasonable view that a statue and a lump of clay might be parts of one another, in a suitably tolerant sense, without being identical. Another difference from Fine is that we do not here take mereological summation or fusion as a primitive "building operation."

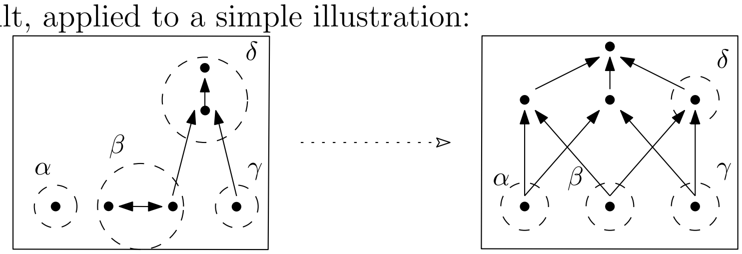
Some of the main informal ideas that motivate our work are these general closure principles: if P is a part-like relation, so is its ancestral, and so is the relation (defined below) we call the *overlap-inclusion* of its ancestral; if P and Q are part-like relations, so is there union. Another general principle that motivates us is that given objects typically bear primitive parthood relations to abstract objects (e.g. sets) "constructed out of" them.

Here is an over-view of what is to come: we introduce the notion of the overlap-inclusion of a given relation, and show that it is intimately connected with the traditional mereological axiom or principle of *strong supplementation*. We then discuss a notion of "sameness of structure" that is weaker than the standard notion of isomorphism, that we call *quasi-isomorphism*. The basic idea is quite simple: structures are quasi-isomorphic if they would be isomorphic when failures of anti-symmetry are "factored out."



three non-isomorphic, quasi-isomorphic reflexive relations

We connect overlap-inclusion to the notions of (mereological) fusion and the axioms of Classical Mereology. Next, we look at some formal structure that concerns the "merging" of relations, which corresponds to the step in the informal presentation above where we went from the separate primitive ways of being a part (R and S) to their union, and then to the ancestral of that union. We show that under a natural general assumption about how "new" things relate to "old" ones, we can show that, when we ignore the differences between quasi-isomorphic structures, there is really only one way to extend the "old" structure: all additions are quasi-isomorphic to a substructure of a certain natural extension of the original, which is itself quasi-isomorphic to a Classical Mereology. The following figure gives a quick rough impression of the result, applied to a simple illustration:



Roughly put, if the structure on the left shows an original domain and some primitive parthood relations on it, then, under a certain main variety of parthood, every "completion" of it is quasi-isomorphic to a substructure of the structure on the right (where the relational links on the right include the transitive closure of those shown).

Most of the arguments below are easy in the sense that they should be accessible to anyone with a basic undergraduate-level background in logic and set theory. The reader should be able to skip any given proof and pick up later, although some forms of argument that are spelled out in detail in one proof will be compressed or left implicit in later proofs.

One reason for a bit of the complexity of the details here is that we attempt to remain fairly neutral about reflexivity. In [Hovda, 2016], the reflexive closure of a part-like relation is taken to be itself part-like, and we do not disagree with that here, but we do not stress it, and our definitions are slightly different, in order to accomodate, as much as possible, any part-like relations there might be that are neither reflexive nor irreflexive.

2 Transitivity

As an introduction to our notation and style, let us consider the general idea of the *ancestral of* or *transitive closure of* a relation. Let R be any relation. Officially, we will work in set theory, and we will think of relations as being sets of ordered pairs. Almost everything we say would be straightforward to adapt to a different background setting, and we may unofficially talk a little differently about relations at times. (In particular, much that we say could be adapted to a setting in which a relation is "too big" to be represented as a *set* of ordered pairs.) As is standard, we will write $x\ R\ y$ to indicate that $\langle x, y\rangle \in R$. And we will sometimes speak of x "bearing R to" y.

Now the ancestral of R, call it $\tau(R)$ or ${}^{\tau}R$ for short, is that relation such that for arbitrary x and y

$x\ {}^{\tau}R\ y$ iff there is a sequence of things $(z_1, \ldots, z_n)$, such that for each $i < n$, $z_i\ R\ z_{i+1}$, and $x = z_1$ and $y = z_n$.

We may indicate such a sequence as a "chain"

$x\ R\ z_2\ R\ z_3\ R \ldots z_{n-1}\ R\ z_n$.

Now it is not hard to see that ${}^{\tau}T$ is a transitive relation, and that it includes R in the sense that if $x\ R\ y$ then $x\ {}^{\tau}R\ y$. It can be shown that ${}^{\tau}T$ is the "smallest" such relation, in the sense that any transitive relation that includes R in this sense also includes ${}^{\tau}T$. Accordingly, we know that if R is itself transitive, then ${}^{\tau}T$ is simply R itself (given, that is, that a relation is just a set of ordered pairs).

As a result, instead of saying that R is transitive, we could simply say $R = {}^{\tau}T$, and we will sometimes do this in the following.

To remind ourselves of our motivation, we conjecture that for every part-like relation R, ${}^{\tau}R$ is also a part-like relation.

3 Overlap-inclusion and strong supplementation

3.1 Overlap

For any relation R, let the "field" of R, or $fld(R)$ for short, be the set of all things x such that there is a y with either $x\ R\ y$ or $y\ R\ x$.

Notational convention: for any relation S, let us write

$x\ S^{=}\ y$

for $x\ S\ y \vee x = y$.

Definition 1 *Given any R, define the relation of (left-) R-overlap as follows:*

$$\forall x \forall y (x \circ_R y \leftrightarrow (x \in fld(R) \wedge y \in fld(R) \wedge \exists z (z\ R^{=}\ x \wedge z\ R^{=}\ y)))$$

We use the word "overlap," and use this notation because of the connection with parthood and standard formal treatments of parthood.

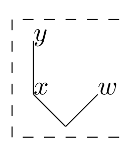
The following four lemmas hold no matter what R, x, and y are:

Lemma 1 *if $x\ R\ y$, then $x \circ_R y$*

Lemma 2 $x \in fld(R)$ *iff* $x \circ_R x$ *iff* $x \in fld(\circ_R)$

Lemma 3 $x \circ_R y$ *iff* $y \circ_R x$

Lemma 4 *If R is transitive, then: if $x\ R\ y$ and $w \circ_R x$, then $w \circ_R y$.*



These are all easy to see; above is a visual "proof" for the last. ⊣

3.2 Overlap-inclusion: general

Definition 2 *Let R be any relation. We define the "(left-)overlap-inclusion" of R, which we will here denote as $\sigma(R)$, or ${}^{\sigma}R$ for short, as follows:*

$$\forall x \forall y (x \, {}^{\sigma}R \, y \leftrightarrow (x \in fld(R) \wedge y \in fld(R) \wedge \forall z (z \circ_R x \rightarrow z \circ_R y))).$$

We will write "$x \, {}^{\sigma}R^{=} \, y$" for "$x \, {}^{\sigma}R \, y \vee x = y$." Note these three facts about overlap-inclusions in general:

Lemma 5 *$x \in fld(R)$ iff $x \, {}^{\sigma}R \, x$ iff $x \in fld({}^{\sigma}R)$*

Lemma 6 *${}^{\sigma}R$ is transitive*

Lemma 7 *if $x \in fld(R)$, then: $x \, {}^{\sigma}R^{=} \, y \rightarrow x \circ_R y$*

Perhaps the third of these is not immediately obvious, but it follows easily once we note that $x \circ_R x$. ⊣

3.3 Overlap-inclusion of a transitive relation

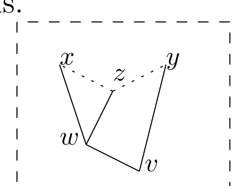
Let T be any relation that is transitive. Then we get a number of nice results about ${}^{\sigma}T$.

Lemma 8 *For $x, y \in fld(T)$: if $x \, T^{=} \, y$ then $x \, {}^{\sigma}T \, y$.*

This is easy to prove by appeal to Lemmas 4 and 5.

Lemma 9 *$x \circ_T y$ iff $x \circ_{{}^{\sigma}T} y$.*

Proof: Left-to-right: get z witnessing $x \circ_T y$; using Lemma 8 it is easy to get that z witnesses $x \circ_{{}^{\sigma}T} y$. Right-to-left: get z witnessing $x \circ_{{}^{\sigma}T} y$. Since $z \, {}^{\sigma}T^{=} \, x$ we get $z \circ_T x$ (recall Lemma 7); let w witness this, so we have $w \, T^{=} \, z$ and $w \, T^{=} \, x$. Now note that since $w \circ_T z$ and $z \, {}^{\sigma}T^{=} \, y$, we may get that $w \circ_T y$; let v witness this.



Now we have $v \, T^{=} \, w$ and $w \, T^{=} \, x$, so, by transitivity of T we have $v \, T^{=} \, x$; but also $v \, T^{=} \, y$, so v witnesses $x \circ_T y$. ⊣

3.4 Strong supplementation

Definition 3 *We will say that a relation Q "obeys strong supplementation" when*

$$(\forall a, b \in fld(Q))\ (\forall x(x \circ_Q a \rightarrow x \circ_Q b) \rightarrow a\ Q\ b)$$

The name "strong supplementation" is standard in the literature; see [Simons, 1987] as well as [Varzi, 2016] and the references therein. (There is also a principle standardly called "weak supplementation," but, despite the names, the two principles are, strictly speaking, independent. The one called "strong" does not imply the "weak" one without further assumptions, not all of which we are making here.)

Theorem 1 *(where T is transitive) ${}^\sigma T$ obeys strong supplementation; i.e., we have that*

$$(\forall a, b \in fld({}^\sigma T))\ \forall x(x \circ_{{}^\sigma T} a \rightarrow x \circ_{{}^\sigma T} b) \rightarrow a\ {}^\sigma T\ b.$$

Proof: Suppose the antecedent. Now, to show that $a\ {}^\sigma T\ b$, we suppose that $x \circ_T a$ and need to show that $x \circ_T b$. Apply Lemma 9 to get $x \circ_{{}^\sigma T} a$, and our hypothesis to get $x \circ_{{}^\sigma T} b$; applying Lemma 9 again gets us $x \circ_T b$. $\dashv$

There is an alternate form of strong supplementation:

Lemma 10 $\forall x(x\ {}^\sigma T\ a \rightarrow x \circ_{{}^\sigma T} b) \rightarrow a\ {}^\sigma T\ b$.

Proof: Left to the reader; do not forget that T is assumed to be transitive. $\dashv$

The ancestral of a transitive relation R is (extensionally) identical with R; we show that a similar result obtains for overlap-inclusion.

Lemma 11 *If T is transitive and obeys strong supplementation, then $a\ T\ b \leftrightarrow a\ {}^\sigma T\ b$.*

Proof: We may assume $a, b \in fld(T)$ (since otherwise we are done by Lemma 5). Assume the hypothesis, so $\forall x(x \circ_T a \rightarrow x \circ_T b) \rightarrow a\ T\ b$. Now $a\ T\ b \rightarrow a\ {}^\sigma T\ b$ is immediate from Lemma 8. And for $a\ {}^\sigma T\ b$ to hold is for $\forall x(x \circ_T a \rightarrow x \circ_T b)$, hence we get $a\ T\ b$ by hypothesis. $\dashv$

Lemma 11 with Theorem 1 obviously yield

Lemma 12 *(for any transitive relation T):*
(A) ${}^\sigma T = \sigma(\sigma(T))$, i.e., $x\ \sigma(T)\ y \leftrightarrow x\ \sigma(\sigma(T))\ y$
(B) T obeys strong supplementation iff $T = {}^\sigma T$.

Hence, in the following, instead of saying "R obeys strong supplementation," we may, when it is known that R is transitive, say simply $R = {}^{\sigma}R$. And note that we may now conclude that for any relation R whatsoever, ${}^{\sigma}({}^{\tau}R)$ obeys strong supplementation, etc. Accordingly, for arbitrary relations Q, let us write ${}^{\Sigma}Q$ for ${}^{\sigma}({}^{\tau}Q)$. Note that we have

${}^{\Sigma}R = R \longleftrightarrow$ (R is transitive and R obeys strong supplementation).

In [Hovda, 2016], an argument is given for the conclusion that (using our notation), if R is a (transitive) part-like relation, then so is ${}^{\sigma}R$. Given the principle that the transitive closure of a part-like relation is itself part-like, this is equivalent to saying that for any part-like R, ${}^{\Sigma}R$ is part-like.

3.5 Reflexivity

While it is common in formal mereology to work with a relation that is taken to be reflexive or irreflexive, we are allowing that a given parthood relation could be neither. (For a brief example motivating this idea, imagine that we take set-theoretic membership to be a variety of parthood, and we also for there to be a set that is a member of itself, as well as one that is not.) This causes some slight deviations in some of our formulations, such as in our definition of overlap above.

Let us look at the situation in a little more detail. Supposing that R is a variety of parthood, one natural but quite strict conception of overlap would say that x overlaps y iff there is a z that bears R to both x and y. But it is also very natural to consider a notion of overlap that ensures that if $x\ R\ y$, then x overlaps y (relative to R), even if x fails to bear R to itself. The notion we have focussed on ensures this, but it is in fact even more permissive, since we have that $x \circ_R x$ iff $x \in fld(R)$. So our notion automatically includes, as cases of x "overlapping" y, two types of case that are not necessarily included on the strictest notion of overlap: the case in which $x\ R\ y$ and the case in which $x = y$. This might seem somewhat artificial, but it is necessary for the way our notion yields nice definitions, of ${}^{\sigma}R$, and of the principle of strong supplementation.

However we define our terms, we are motivated by the thought that if R is a variety of parthood, then so is (the relation we have called) ${}^{\Sigma}R$, and, remarkably, this relation is guaranteed to satisfy a principle very much like that of the classical strong supplementation axiom; our definitions have the virtue of letting us state all this fairly smoothly. Whatever you call it, the relation ${}^{\Sigma}R$ is, we want to say, is a parthood-like relation, though a tolerant (rather than strict), and perhaps derivative (rather than primitive) one.

In fact, however, there may be reason to consider a broader conception of a tolerant, derivative form of parthood, based on a given form of parthood R. For it may be that R, roughly speaking, naturally "goes with" a certain domain D that is in fact a proper superset of $fld(R)$. For one example, on some views, (primitive) parthood among concrete objects is a relation that runs from part x to concrete whole y only when x is among some things that collectively behave in a unified manner (and these will be the parts of y). On this sort of view, if b is a concrete thing that has no parts, and does not join with any other things in unified behavior, then b is not part of anything. Yet, suppose that partless z is primitively a part of something; in this case, given our definitions, and denoting primitive concrete parthood with R, z will bear ${}^{\Sigma}R$ to itself, but b will not bear ${}^{\Sigma}R$ to itself. We should consider the relation Q that x bears to y just when either $x\ {}^{\Sigma}R\ y$ or $x = y$ and x is a concrete thing. There is a sense in which primitive concrete parthood is "appropriate" for object b even though b is not in its field, while it is not "appropriate" for an abstract thing like the number 3, which is also not in its field. One way to reflect this is to regard the relation Q as a variety of parthood, while denying that the relation $Q \cup \langle 3, 3\rangle$ is a variety of parthood, or at least holding that the former is more natural than the latter. In fact, Q may be regarded as more natural than ${}^{\Sigma}R$.

In the following, we will continue to make a certain amount of room for relations that are neither reflexive nor irreflexive, but we will also bear in mind situations like the example of the R and Q just discussed. In general form, we have a domain D, a relation R whose field is a subset of D and we wish to speak of the relation that results from "expanding" ${}^{\Sigma}R$ to include all the $\langle x, x\rangle$ pairs where $x \in D$. We could achieve this effect in a number of equivalent ways. To illustrate, let $=_D$ be the relation of identity restricted to D, so that our target is $({}^{\Sigma}R) \cup =_D$. There are many ways we could reach this target. It can be shown that

$${}^{\tau}(R) \cup =_D \quad = \quad {}^{\tau}(R \cup =_D)$$

and in fact

$$({}^{\Sigma}R) \cup =_D \quad = \quad {}^{\sigma}({}^{\tau}(R)) \cup =_D \quad = \quad {}^{\sigma}({}^{\tau}(R \cup =_D)) \quad = \quad {}^{\sigma}({}^{\tau}(R) \cup =_D).$$

Thus $=_D$ can be "added," so to speak, at any of a few stages along the way from our given parthood relation R, to our eventual target. In the example recently discussed it might well be thought that (the correlate of) $R \cup =_D$ is itself a natural part-like relation, since we might say that each concrete thing itself behaves in a unified manner, hence, in a limit-case way, is automatically among some "things" (namely it and itself) that together behave in a unified manner. Hence there is some justification for looking at situations where we have a relation R that is reflexive on a given associated domain D (i.e. $x\ R\ x$ for all $x \in D$). This is effectively what we will do in the final section of the paper.

3.6 Fusion

We will define a notion of fusion that is basically standard, but (as is less familiar) explicitly relativized to a relation, and which is slightly more tolerant of irreflexive or non-reflexive relations.

Definition 4 *Define, for any relation Q*

$$Fu_Q(b, [x|\phi_x])$$

as

$$\forall x\,(\phi_x \to x\ Q^{=}\ b) \wedge \forall w\,(w\ Q^{=}\ b \to \exists x\,(\phi_x \wedge x \circ_Q w)).$$

We formulate our definition schematically, with the idea that ϕ_x could be any formula of any language whatsoever (provided that language includes the logical apparatus used here, and a relation-symbol Q, or suitable surrogates). If we wished to be more rigorous, of course, we could specify some formal language instead. As an abbreviation device, when it is clear that X is a set, we will write $Fu_Q(b, X)$ to mean $Fu_Q(b, [x|x \in X])$.

Lemma 13 *If $Fu_Q(b, [x|\phi_x])$, then:*
(A) $\exists x\, \phi_x$
(B) $b \in fld(Q)$
(C) $\forall x\,(\phi_x \to x \in fld(Q))$.

Proof: Straightforward; note that $b\ Q^{=}\ b$ by logic alone, and $fld(Q) = fld(\circ_Q)$. $\dashv$

Lemma 14 *If $T =^{\tau} T$ then:*
(A) $Fu_T(b, [x|\phi_x]) \to Fu_{^\sigma T}(b, [x|\phi_x])$
(B) $Fu_T(b, [x|\phi_x]) \leftrightarrow$
$[b \in fld(T) \wedge \forall x\,(\phi_x \to x\ T^{=}\ b) \wedge \forall w\,(w \circ_T b \to \exists x\,(\phi_x \wedge x \circ_T w))]$
(C) $Fu_{^\sigma T}(b, [x|\phi_x]) \leftrightarrow$

$[b \in fld(T) \wedge \forall w\,(w \circ_{\sigma T} b \leftrightarrow \exists x\,(\phi_x \wedge x \circ_{\sigma T} w))]$
(D) if both $Fu_{\sigma T}(b, [x|\phi_x])$ *and* $Fu_{\sigma T}(b', [x|\phi_x])$*, then both* $b\ {}^{\sigma}T\ b'$ *and* $b'\ {}^{\sigma}T\ b$
(E) if both $Fu_T(b, [x|\phi_x])$ *and* $Fu_T(b', [x|\phi_x])$*, then both* $b\ {}^{\sigma}T\ b'$ *and* $b'\ {}^{\sigma}T\ b$
(F) if $b \in fld(T)$*, then* $Fu_T(b, [x|x\ T^{=}\ b])$
(G) if both $b\ {}^{\sigma}T\ b'$ *and* $b'\ {}^{\sigma}T\ b$*, and also* $Fu_{\sigma T}(b, [x|\phi_x])$*, then* $Fu_{\sigma T}(b', [x|\phi_x])$*.*

Proof: Now for (A), suppose $\forall x\,(\phi_x \to x\ T^{=}\ b) \wedge \forall w\,(w\ T^{=}\ b \to \exists x\,(\phi_x \wedge x \circ_T w))$. Then $\forall x\,(\phi_x \to x\ {}^{\sigma}T^{=}\ b)$ is immediate from Lemma 8 (and the fact that $fld(T) = fld({}^{\sigma}T)$). Now suppose that $w\ {}^{\sigma}T^{=}\ b$; we need to show that $\exists x\,(\phi_x \wedge x \circ_{\sigma T} w))$. By Lemma 7, $w \circ_T b$; let z witness this, so $z\ T^{=}\ w$ and $z\ T^{=}\ b$. The latter and our hypothesis get us an $x \in G$ with $(\phi_x \wedge x \circ_T z))$. Apply Lemma 4 and conclude $x \circ_T w$; then Lemma 9 yields $x \circ_{\sigma T} w$, and we have (A). (B) is clear from applications of Lemmas 1, 2, and 4.

For (C), the left-to-right direction is similar to (B). For the right-to-left: supposing the right side, we first show that $\forall x\,(\phi_x \to x\ {}^{\sigma}T^{=}\ b)$. Now if we have an x with ϕ_x, we get that for any z, if $z \circ_{\sigma T} x$ then, by supposition $z \circ_{\sigma T} b$. But Theorem 1 then yields $x\ {}^{\sigma}T\ b$. And it is easy to confirm the other fusion condition, so we have $Fu_{\sigma T}(b, [x|\phi_x])$.

To get (D): supposing the hypothesis and that $z \circ_T b$, we want to show that $z \circ_T b'$. We get $z \circ_{\sigma T} b$, and we may then activate the fusion condition on b (and, in view of (B)) get an x such that ϕ_x and $z \circ_{\sigma T} x$ Given the fusion condition on b', and (C), we have $z \circ_{\sigma T} b'$, so $z \circ_T b'$, as desired. Hence $b\ {}^{\sigma}T\ b'$ and the same reasoning yields $b'\ {}^{\sigma}T\ b$. (E) is immediate from (A) and (D). (F) and (G) are easy to confirm. $\dashv$

Note that parts (B) and (C) show us, in effect, alternative ways to define fusion (given background assumptions of transitivity or obedience of strong supplementation). See [Hovda, 2009] and [Varzi, 2016] for discussion of various definitions of fusion.

Definition 5 *For any relation* R *and sets* A *and* B*, write*
$A \cong_{\circ_R} B$
for $A, B \subseteq fld(R)$*, and* $\forall x((\exists y \in A)\, x \circ_R y \ \leftrightarrow\ (\exists y \in B)\, x \circ_R y)$*.*

Lemma 15 *Suppose* $S = {}^{\Sigma}S$ *and that* A *and* B *are sets. Then*
$Fu_S(b, A) \ \to\ (Fu_S(b, B) \leftrightarrow A \cong_{\circ_S} B)$*.*

Proof: Supposing $Fu_S(b, A)$, the left-to-right direction is easy (cf. Lemma 14(C)). For the right-to-left, assume $A \cong_{\circ_S} B$ and let $x \in B$. We will argue that $x\ S\ b$ by way of strong supplementation (noting that $x, b \in fld(S)$). Suppose $z \circ_S x$; by hypothesis, get $y \in A$ with $z \circ_S y$, hence $z \circ_S b$, and we thus deduce $x\ S\ b$. Next, we need to show that if $z\ S^{=}\ b$, then $(\exists y \in B)\ z \circ_S b$. But this is straightforward from our hypotheses. $\dashv$

4 Quasi-isomorphism

Later, we will show that under certain general and natural assumptions, there is a sense in which our principles for finding new part-like relations, given old ones, are guaranteed to stop generating any novel structure after only a few applications. Roughly, a certain structure arises quickly, as a "fixed point." To make this more precise, we will use a notion of "quasi-isomorphism," which we will now define and investigate several formal features of, especially in connection to part-like relations and the axioms of Classical Mereology.

First, let us articulate the standard idea of an isomorphism between two relational structures:

> Let R and S be relations whose fields are subsets of D, and E, respectively. Then f is an *isomorphism* from $\langle D, R\rangle$ to $\langle E, S\rangle$ iff
>
> > $f : D \to E$ (i.e., f is a function with domain D and whose range is a subset of E); and
> > f is one-to-one (i.e. $f(x) = f(y)\ \to\ x = y$); and
> > f is onto E (i.e. $(\forall y \in E)(\exists x \in D)\, y = f(x)$); and
> > $(\forall x, y \in D)\ (x\ R\ y \leftrightarrow f(x)\ S\ f(y))$.

We say that two structures are isomorphic when there exists an isomorphism from one to the other. Such pairs of structures are "structurally identical;" R is transitive iff S is transitive, etc.

Now, the intuitive version of our notion of quasi-isomorphism is this: suppose we are considering relations R and S that are transitive. Now if, for some x, y with $x \neq y$, $x\ R\ y$ and $y\ R\ x$, then x and y are "equivalent" relative to R: in particular, $x\ R\ a \leftrightarrow y\ R\ a$ and $a\ R\ x \leftrightarrow a\ R\ y$, no matter what a is. Now it could be that $\langle D, R\rangle$ and $\langle E, S\rangle$ are not isomorphic, but "would be if we identify equivalent things." This is roughly what we mean when we say that they are "quasi-isomorphic."

To make this more precise, we will first define the idea of a "reduction" of a given structure, which will be a substructure of it, that it is quasi-isomorphic to, and which is quasi-isomorphic to no "smaller" substructure. The reduction will basically just choose one out of every collection of equivalent things, and throw the rest of the things out, and leave the remaining relational links intact. (At times it would have been easier to work with new relations relating equivalence-classes of our old things, rather than working with restrictions of the old relations, but, for our eventual application, we wish our approach to be more directly adaptable to situations in which the equivalence-classes in question would be "too big to be a set.")

First, a couple abbreviations:

we will write $x \equiv_R y$ to mean that $(x\ R\ y \wedge y\ R\ x)$;

we will write $x \equiv_R^= y$ to mean that $(x \equiv_R y \vee x = y)$.

Definition 6 *$\langle D^-, R^-\rangle$ is a substructure of $\langle D, R\rangle =_{def} D^- \subseteq D$ and $R^- = R \restriction D^-$ (i.e., $(\forall x, y)(x\ R^-\ y \leftrightarrow (x \in D^- \wedge y \in D^- \wedge x\ R\ y)))$.*

Now for our next notions:

Definition 7 *Say that $c : D \to D$ "reduces" $\langle D, R\rangle$ (to $\langle D^-, R^-\rangle$) iff R is a relation with $fld(R) \subseteq D$, meeting these conditions:*

(i) $(\forall x \in D)\ x \equiv_R^= c(x)$ and

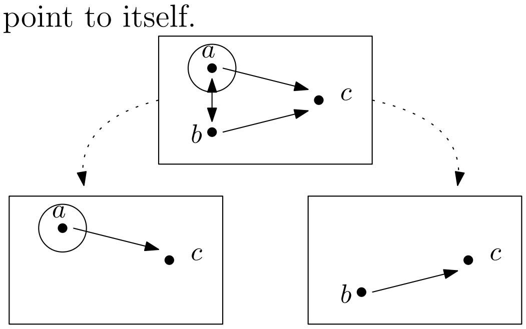
(ii) $(\forall x, y \in D)\,(x \equiv_R^= y\ \leftrightarrow\ c(x) = c(y))$ and

(iii) $D^- = ran(c)$ and $R^- = R \restriction D^-$.

Say that $\langle D^-, R^-\rangle$ is a "reduction" of $\langle D, R\rangle$ if there is a c that reduces $\langle D, R\rangle$ to $\langle D^-, R^-\rangle$.

Say that $\langle D, R\rangle$ is "quasi-isomorphic" to $\langle E, S\rangle$ if there is a reduction of the one that is isomorphic to a reduction of the other.

Now, this definition makes quasi-isomorphism non-transitive. For example, consider the structures pictured below, where a solid arrow indicates a holding of the relation, and a circle around a point indicates the holding of the relation from that point to itself.



The structure at the top may be reduced, by our definition, to either of the ones below. But those two are not isomorphic, yet each reduces to itself, and hence would be quasi-isomorphic to itself, so we would have a failure of transitivity of "quasi-isomorphism".

Yet this kind of situation will only arise with relations that are not transitive, as we will show with Theorem 2.

Lemma 16 *Let c reduce $\langle D, R\rangle$ to $\langle D^-, R^-\rangle$, and let $a, b \in D^-$. Then:*
(A) $a\ R\ b \leftrightarrow a\ R^-\ b$
(B) $a = c(a)$
(C) R^- is anti-symmetric (i.e., $(a\ R^-\ b \wedge b\ R^-\ a) \rightarrow a = b$)
(D) if R is anti-symmetric, then $(\forall x \in D)\, c(x) = x$.

Proof: (A) is obvious. For (B), if $a = c(x)$, then, since $x \equiv_R^= a$, $c(x) = c(a)$. For (C), if $(a\ R^-\ b \wedge b\ R^-\ a)$, then $a \equiv_R^= b$ so $c(a) = c(b)$, but then, applying (B), we have $a = c(a) = c(b) = b$. For (D), we are supposing that $\forall x, y\, (x \equiv_R y \rightarrow x = y)$. So $(x \equiv_R^= y \rightarrow x = y)$. Now for $x \in D$, $x \equiv_R^= c(x)$, so $x = c(x)$. $\dashv$

From (D) we see that if R is anti-symmetric, then $\langle D, R\rangle$ is "irreducible" in the sense that the one and only reduction of it is the identity map. Then with (C) we see that for any $\langle D, R\rangle$ whatsoever, every reduction of it is irreducible.

4.1 Reduction of a transitive relation

We have seen some properties of reductions that do not depend on transitivity, but our main use of the notion is for transitive relations. We first note that for these, we may state the condition of something's being a reduction in somewhat simpler terms:

Lemma 17 *Suppose $R =^\tau R$ (i.e., R is transitive), with $fld(R) \subseteq D$. Then $c : D \rightarrow D$ reduces $\langle D, R\rangle$ iff*
(i) $(\forall x \in D)\, x \equiv_R^= c(x)$ and
(ii) $(\forall x, y \in D)\, (x \equiv_R^= y \ \rightarrow\ c(x) = c(y))$.

Proof: Given transitivity of R and that condition (i) is met, we may easily verify that $c(x) = c(y) \quad \rightarrow \quad x \equiv_R^= y$, since then we have $x \equiv_R^= c(x) \equiv_R^= c(y) \equiv_R^= y$. $\dashv$

Lemma 18 *Suppose $R =^\tau R$, and let c reduce $\langle D, R\rangle$, to $\langle D^-, R^-\rangle$. Then:*
(A) R^- is transitive
(B) $(\forall x, y \in D): \; x \circ_R y \;\leftrightarrow\; c(x) \circ_R y$
(C) $(\forall x, y \in D): \; x \; R \; y \;\leftrightarrow\; c(x) \; R \; c(y) \;\leftrightarrow\; c(x) \; R^- \; c(y)$
(D) $(\forall a, b \in D^-): \; a \circ_R b \;\leftrightarrow\; a \circ_{R^-} b$
(E) $(\forall x \in D)(\forall a \in D^-): \; x \circ_R a \;\leftrightarrow\; c(x) \circ_{R^-} a$
(F.1) $c(x) \; R^- \; c(x) \;\leftrightarrow\; x \; R \; x \;\leftrightarrow\; (\exists y \in D) \; x \equiv_R y$
(F.2) if either $\neg \, x \; R \; x$ or $\neg \, c(x) \; R^- \; c(x)$, then $c(x) = x$
(G) $fld(R^-) \subseteq fld(R)$, and $(\forall x \in D) \, (x \in fld(R) \;\leftrightarrow\; c(x) \in fld(R^-))$.

Proof: Note that if $\alpha \; R^= \; \beta$ and $\beta \; R^= \; \gamma$ then $\alpha \; R^= \; \gamma$. Now (A) is obvious, given what R^- is. (B) is easy (recalling Lemma 4) since $x \; R^= \; c(x)$ and vice-versa. For the first biconditional of (C), if $x \; R \; y$, we have $c(x) \; R^= \; x \; R \; y \; R^= \; c(y)$; and if $c(x) \; R \; c(y)$, we have $x \; R^= \; c(x) \; R \; c(y) \; R^= \; y$. The second biconditional of (C) is trivial. For (D), left-to-right: suppose $z \; R^= \; a$ and $z \; R^= \; b$ and $a, b \in fld(R)$; then $a, b \in fld(R^-)$ (cf. part (G)), and we have $c(z) \; R^= \; z \; R^= \; a$ and $c(z) \; R^= \; z \; R^= \; b$. So $c(z) \; R^{-=} \; a$ and $c(z) \; R^{-=} \; b$, as desired. The right-to-left part of (D) is easy. (E) can be obtained easily from (B) and (D). The first biconditional in (F.1) is easy from (C), and the second biconditional is elementary. (F.2) is easy and (G) is straightforward, recalling (C). ⊣

(C) makes explicit the way the reduction of $\langle D, R\rangle$ "represents" the structure of R, and helps to justify our term "quasi-isomorphic," as does

Theorem 2 *Suppose R is transitive, and let c reduce $\langle D, R\rangle$, to $\langle D^-, R^-\rangle$. Then every reduction of $\langle D, R\rangle$ is isomorphic to $\langle D^-, R^-\rangle$.*

Proof: Suppose that d reduces $\langle D, R\rangle$ to $\langle D_-, R_-\rangle$; we show that $g = d \restriction D^-$ is an isomorphism from $\langle D^-, R^-\rangle$ to $\langle D_-, R_-\rangle$. (g is the function with domain D^- such that for all $x \in D^-$, $g(x) = d(x)$.) Now $(\forall x, y \in D^-) \; (x \neq y \rightarrow \neg \, x \equiv_R^= y)$, so g is one-to-one. And if $y \in D_-$, $c(y) \in D^-$, and $g(c(y)) \in D_-$. But we also have $y \equiv_R^= c(y) \equiv_R^= g(c(y))$, so $y = g(c(y))$; hence g is onto D_-. And, appealing to Lemma 18(C) (applied to d), we see that for any $x, y \in D^-$, $x \; R^- \; y \leftrightarrow x \; R \; y \leftrightarrow g(x) \; R_- \; g(y)$. ⊣

An obvious corollary is

Lemma 19 *If $\langle D, R\rangle$ is quasi-isomorphic to $\langle E, S\rangle$, and both R and S are transitive, then every reduction of the one is isomorphic to every reduction of the other.*

4.2 Features preserved across reductions

Lemma 20 *Suppose* $R =^{\tau} R$ *and that* c *reduces* $\langle D, R\rangle$, *to* $\langle D^-, R^-\rangle$. *Then:*
$R = {}^{\sigma} R \; \leftrightarrow \; R^- = {}^{\sigma}(R^-)$

Proof: We will make repeated appeal to parts of Lemma 18, which we will refer to by their letters. For left-to-right: we are supposing that $(\forall x, y \in fld(R))(\forall z(z \circ_R x \rightarrow z \circ_R y) \rightarrow x \; R \; y)$. Now suppose that for some $a, b \in fld(R^-)$, we have that $\forall z(z \circ_{R^-} a \rightarrow z \circ_{R^-} b)$; we must show that $a \; R^- \; b$. By (G), $a, b \in fld(R)$, and so, by our main supposition, we will be done if we can show that $\forall z(z \circ_R a \rightarrow z \circ_R b)$. Now if $z \circ_R a$, then $z \in fld(R)$, and so by (E), $c(z) \circ_{R^-} a$. So, by supposition, $c(z) \circ_{R^-} b$. But now by (D), $c(z) \circ_R b$, so by (B), $z \circ_R b$ and we are done.

For right-to-left: we are supposing that $(\forall a, b \in fld(R^-))(\forall z(z \circ_{R^-} a \rightarrow z \circ_{R^-} b) \rightarrow a \; R^- \; b)$. Now suppose that for some $x, y \in fld(R)$, we have that $\forall z(z \circ_R x \rightarrow z \circ_R y)$; we must show that $x \; R \; y$. Now we know, by (G), that $c(x), c(y) \in fld(R^-)$. Suppose now that $z \circ_{R^-} c(x)$. Then $z \in fld(R^-) \subseteq D^-$ and so $z \circ_R c(x)$, and so $z \circ_R x$. Now, by supposition, $z \circ_R y$, so $z \circ_R c(y)$, and so $z \circ_{R^-} c(y)$. z was arbitrary, so, by our main supposition, $c(x) \; R^- \; c(y)$, and hence $x \; R \; y$ and we are done. $\dashv$

4.2.1 Fusions and reduction

Recall our abbreviation device that for a set X $Fu_Q(b, X)$ means $Fu_Q(b, [x|x \in X])$.

Lemma 21 *Suppose* $R = {}^{\tau} R$, *and let* c *reduce* $\langle D, R\rangle$, *to* $\langle D^-, R^-\rangle$. *Suppose* $X \subseteq D$, $Y \subseteq D^-$, $(\forall x \in X) \; c(x) \in Y$ *and* $(\forall y \in Y)(\exists x \in X) \; c(x) = y$. *Then:*
$Fu_R(b, X) \; \leftrightarrow \; Fu_{R^-}(c(b), Y)$

Proof: Left-to-right: if $y \in Y$, get $x \in X$ with $c(x) = y$, so $x \; R^= \; b$ by hypothesis, so $c(x) \; R^{-=} \; c(b)$, recalling Lemma 18(C). For the second fusion condition, note $b \in fld(R)$, and confirm $c(b) \in fld(R^-)$, recalling Lemma 18(G). Now, bearing in mind Lemma 14(B), we argue: if $z \circ_{R^-} c(b)$, then note that $z \in D^-$, and we have $z \circ_R c(b)$, so $z \circ_R b$, hence there is an $x \in X$ with $z \circ_R x$; but then $z \circ_{R^-} c(x)$. So we have that if $z \circ_{R^-} c(b)$, then $(\exists y \in Y) \; z \circ_{R^-} y$, as desired.

For the other direction, similarly: if $x \in X$, $c(x) \in Y$, so $c(x) \; R^{-=} \; c(b)$, so $x \; R^= \; b$. Note $c(b) \in fld(R^-)$ so $b \in fld(R)$. And, supposing $z \circ_R b$, we deduce $c(z) \circ_{R^-} c(b)$, hence there is a $y \in Y$ with $c(z) \circ_{R^-} y$. But then there is an $x \in X$ with $c(x) = y$, and we get, appealing to Lemma 18(E), that $c(z) \circ_R x$ and hence $z \circ_R x$ with $x \in X$, as desired. $\dashv$

4.2.2 Fusion-completeness

Definition 8 *D is R-fusion complete* $=_{def}$
D is non-empty and, for every non-empty $X \subseteq D$, $(\exists b \in D)\ Fu_R(b, X)$.

Lemma 22 *If D is R-fusion complete, then* $D \subseteq fld(R)$.

Proof: Given $z \in D$, get $b \in D$ with $Fu_R(b, [x|x = z])$, deduce $\exists x(x = z \wedge x \circ_R z)$, hence $z \circ_R z$, so $z \in fld(R)$. $\dashv$

Lemma 23 *Suppose* $R = {}^\tau R$, *and let c reduce* $\langle D, R\rangle$, *to* $\langle D^-, R^-\rangle$. *Then:*
D is R-fusion complete iff D^- *is* R^-*-fusion complete.*

Proof: Left-to-right: if Y is a non-empty subset of D^-, then, by hypothesis, there is some $b \in D$ such that $Fu_R(b, Y)$. But Lemma 21 yields $Fu_{R^-}(c(b), Y)$. For the other direction: for any non-empty $X \subseteq D$, let Y be $\{y \in D^- : y = c(x) \text{ for some } x \in X\}$. By hypothesis, there is a $b \in D^-$ with $Fu_{R^-}(b, Y)$. Since $b = c(b)$, we may deduce, using Lemma 21 again, $Fu_R^D(b, X)$. $\dashv$

Lemma 24 *Suppose that R and S are transitive and that* $\langle D, R\rangle$ *is quasi-isomorphic to* $\langle E, S\rangle$. *Then:*
(A) ${}^\sigma R = R$ *iff* ${}^\sigma S = S$
(B) D is R-fusion complete iff E is S-fusion complete.

Proof: Get reductions $\langle D^-, R^-\rangle$ and $\langle E^-, S^-\rangle$ of $\langle D, R\rangle$ and $\langle E, S\rangle$. By Lemma 19, $\langle D^-, R^-\rangle$ and $\langle E^-, S^-\rangle$ are isomorphic. Hence, if one of these obeys strong supplementation, so does the other; similarly for fusion completeness. (We leave it to the reader to confirm this elementary result.) Now apply Lemmas 20 and 23. $\dashv$

4.3 The existence of reductions and quasi-isomorphisms

Next we note that reductions exist, in general, for our desired application:

Lemma 25 *Let R be a transitive relation with $fld(R) \subseteq D$. Then there exists a function $c : D \to D$ that reduces $\langle D, R\rangle$.*

Proof: For each $x \in D$, let $[x]_R = \{y \in D : y \equiv_R^= x\}$. $X \subseteq D$ is an equivalence-class under $\equiv_R$ iff there is $x \in D$ such that $X = [x]_R$. By the axiom of choice, there is a function e whose domain is the set of these equivalence-classes and such that for every such class X, $e(X) \in X$. Now let $c : D \to D$ be the function $c(x) = e([x]_R)$ and it is easy to verify that c reduces $\langle D, R\rangle$. (We need transitivity of R to ensure that if $x \equiv_R^= y$ then $[x]_R = [y]_R$.) $\dashv$

Theorem 2 and Lemma 25 tell us that for each transitive relational structure, there are reductions of it, but they are unique "up to isomorphism." In the language of isomorphism types, each transitive relational structure has a unique reduction type. The reduction type is (of) a transitive, anti-symmetric relation.

Lemma 26 *Suppose $\langle C, Q\rangle$ is a substructure of $\langle D, R\rangle$ and $R =^\tau R$, and $fld(R) \subseteq D$. Suppose $(\forall x \in D)(\exists y \in C)\ x \equiv_R^= y$. Then $\langle D, R\rangle$ is quasi-isomorphic to $\langle C, Q\rangle$, and in particular, there is a $\langle C^-, Q^-\rangle$ that is a reduction of both.*

Proof: By Lemma 25, get a $c : C \to C$ that reduces $\langle C, Q\rangle$ to $\langle C^-, Q^-\rangle$. Now, since for each $x \in D$ there is at least one $y \in C$ with $x \equiv_R^= y$, we may use the axiom of choice to obtain a function $d : D \to C$ that chooses one of these; i.e., we have $(\forall x \in D)\ d(x) \equiv_R^= x$. Then set $b : D \to C$ as $b(x) = c(d(x))$ and it is easy to show that b reduces $\langle D, R\rangle$ to $\langle C^-, Q^-\rangle$. $\dashv$

4.4 Unique fusions and Classical Mereology

Definition 9 *Say that D is R-fusion !-complete iff:*
D is non-empty and for every non-empty $X \subseteq D$, $(\exists! b \in D)\ Fu_R(b, X)$

Lemma 27 *Suppose $R = {}^\Sigma R$, $fld(R) \subseteq D$, and that D is R-fusion complete. Then (by Lemma 22) we have $fld(R) = D$, and:*
(A) R is anti-symmetric iff D is R-fusion !-complete
(B) if c reduces $\langle D, R\rangle$, to $\langle D^-, R^-\rangle$, then D^- is R^--fusion !-complete.

Proof: For (A), the left-to-right direction is straightforward from Lemma 14(D) and the anti-symmetry of ${}^{\Sigma}R$ ($={}^{\sigma}R$). The right-to-left direction may be obtained when we observe (Lemma 14(F)) that under the supposed conditions every $y \in D$ is an R-fusion of the things $R^{=}\ y$, i.e., $Fu_R(y, [x|xR^{=}y])$. For it is easy to see that if $y \equiv^R z$, then also $Fu_R(z, [x|xR^{=}y])$. For (B), combine Lemma 23 with Lemma 16(C).⊣

Structures with a transitive relation R that obeys strong supplementation and where $fld(R)$ is R-fusion !-complete are Classical Mereologies in the sense that the relation R on the domain $fld(R)$ obeys all the axioms of Classical Mereology, interpreting "parthood" or "$\leq$" as R, and where we assume that the fusion-existence axiom (or axiom scheme) is formulated in such a way as to guarantee the actual "completeness" of the structure: basically, for *every* non-empty subset of the domain, there is a fusion for that subset (not just for the subsets that are expressed by an open formula of some language). ([Hovda, 2009] has an extended discussion of the axioms of Classical Mereology and a brief discussion of the issue of how to ensure the "completeness" in question.) The characteristic features are that parthood is transitive and every non-empty condition has a unique fusion. And so we have that if R is transitive, obeys strong supplementation, and is fusion complete on $D = fld(R)$, then $\langle D, R\rangle$ reduces to (and is thus quasi-isomorphic with) a Classical Mereology.

Let us call a structure $\langle D, R\rangle$ a *sub-Classical Mereology* if $D = fld(R)$, R obeys the axioms of transitivity, strong supplementation, and D is R-fusion-complete. (We ignore here the issue of how fusion-completeness would be expressed axiomatically.) Then we may restate some of the content of Lemma 27 as

Theorem 3 *Every sub-Classical Mereology reduces to (and hence is quasi-isomorphic with) a Classical Mereology.*

This is basically to say that if we add anti-symmetry to the axioms for sub-Classical Mereology, we get Classical Mereology.

4.5 Groundedness

We will now show a number of results that involve situations in which we have a set B that is "grounded" in a set A in the sense that every $y \in B$ has something in A that bears some relevant relation to y.

Definition 10 *For any set C and relation R, for any y:*
$[R]\,y$ *(y's left R-extension)* $=_{def} \{x : x\ R\ y\}$ *(assuming this set exists)*
$[R^{=}]\,y =_{def} \{x : x\ R^{=}\ y\}$ *(again, assuming it exists)*
$[R_C]\,y$ *(y's R-extension in C)* $=_{def} \{x \in C : x\ R\ y\}$
$[R_C^{=}]\,y =_{def} \{x \in C : x\ R^{=}\ y\}$.

(The qualifications about existence are only relevant if we are working with a relation R that is itself "too big" to be a set. In most of our development, we are officially ignoring such relations, but we do wish many of our results to extend to such relations.)

Definition 11 *For any sets C, D and relation R, say that D is R-grounded in C when:*
$(\forall y \in D)\ [R_C]\,y \neq \emptyset$, *i.e.,* $(\forall y \in D)(\exists x \in C)\ x\ R\ y$.
Say that D is $R^{=}$-grounded in C when
$(\forall y \in D)\ [R_C^{=}]\,y \neq \emptyset$, *i.e.,* $(\forall y \in D)(\exists x \in C)\ x\ R^{=}\ y$.

Lemma 28 *If $R = {}^{\tau}R$ and $x\ R\ y$, then:*
(A) $[R]\,x \subseteq [R]\,y$, $[R^{=}]\,x \subseteq [R^{=}]\,y$, $[R_C]\,x \subseteq [R_C]\,y$, *and* $[R_C^{=}]\,x \subseteq [R_C^{=}]\,y$;
(B) if, in addition, either $[R^{=}]\,x$ *or* $[R^{=}]\,y$ *(exists and) is $R^{=}$-grounded in C, and $a\ R^{=}\ x$ and $b\ R^{=}\ y$, then, if $a \circ_R b$, there is a $w \in C$ that witnesses the overlap, i.e., $w\ R^{=}\ a$ and $w\ R^{=}\ b$.*

Proof: Part (A) is obvious; for part (B), observe that we have a z such that $z\ R^{=}\ a$ and $z\ R^{=}\ b$; but then $z\ R^{=}\ x$ and $z\ R^{=}\ y$, hence, one way or the other, there is a $w \in C$ with $w\ R^{=}\ z$. But then w is the desired witness. $\dashv$

Lemma 29 *If $R = {}^{\Sigma}R$ then:*
(A) $x\ R\ y \longleftrightarrow [R]\,x \subseteq [R]\,y$
(B) if, in addition, $[R]\,x$ *is $R^{=}$-grounded in C, then*

(B.1) $x\ R\ y \longleftrightarrow [R_C]\,x \subseteq [R_C]\,y$
(B.2) $x\ R\ y \longleftrightarrow (\forall z \in [R_C]\,x)(\exists w \in [R^{=}]y)\ z \circ_R w$
(B.3) $x\ R\ y \longleftrightarrow (\forall z \in [R_C]\,x)(\exists w \in [{R_C}^{=}]y)\ z \circ_R w$
(B.4) $x\ R\ y \longleftrightarrow (\forall z \in [R_C]\,x)\ x \circ_R y$.

Proof: The left-to-right directions are covered by Lemma 28(A) or are easy. For the right-to-left direction of (A), check that if $a \; R \; x$, then $a \; R \; y$, hence $a \circ_R y$; hence (cf. Lemma 10) $x \; R \; y$. For the right-to-left direction of (B.2): if $a \; R \; x$, then, by hypothesis, there is a $b \in C$ with $b \; R \; a$. Then $b \in [R_C]x$, so, by our other hypothesis, we get $w \in [R^{=}] \, y$ with $b \circ_R w$. We also have $w \; R^{=} \; y$, $b \; R \; a$, hence $a \circ_R y$. Apply Lemma 10. The right-to-left directions of all the others follow easily from (B.2). $\dashv$

Theorem 4 *Suppose* $R = {}^{\Sigma}R$, $fld(R) \subseteq D$, $C \subseteq D$ *and let* $Q = R \restriction C$ *and suppose that* C *is* Q*-fusion !-complete and that* D *is* $R^{=}$*-grounded in* C*. Then:*

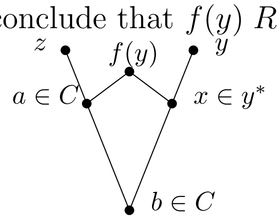
(A) $\langle C, Q \rangle$ *is a reduction of* $\langle D, R \rangle$

(B) $\langle C, Q \rangle$ *is a Classical Mereology.*

Proof: First, note that for any $y \in D$, by groundedness, there is a $z \in C$ such that $z \; R^{=} \; y$; then, recalling Lemma 22, we have $z \in fld(Q) \subseteq fld(R)$, from which we may deduce that $y \in fld(R)$. So $D = fld(R)$.

Now, for a notion we will use repeatedly below, for $y \in D$, let $y^* = [R^{=}_{C}] \, y$, i.e., $\{a \in C : a \; R^{=} \; y\}$. Now let $f(y)$ be the unique $b \in C$ such that $fu_Q(b, [x | x \in y^*])$. By Lemma 28, if $z \; R^{=} \; y$ then $z^* \subseteq y^*$.

Now we claim that (i) for all $y \in D$, $y \equiv_R f(y)$. First, suppose that $z \circ_R f(y)$; we will argue that $z \circ_R y$ and use strong supplementation (on R, whose field includes D) to conclude that $f(y) \; R \; y$.



By Lemma 28(B), get $a \in C$ with $a \; R^{=} \; z$ and $a \; R^{=} \; f(y)$. Given the fusion condition on $f(y)$, get $x \in y^*$ with $a \circ_Q x$; let $b \in C$ witness this. It should be clear from the figure now that $z \circ_R y$. For the second half of (i), we argue that if $z \circ_R y$, then $z \circ_R f(y)$ and use strong supplementation to conclude that $y \; R \; f(y)$. Let w witness $z \circ_R y$ and by Lemma 28(A) $w^* \subseteq y^*$. By groundedness, get $x \in w^*$, hence $x \in y^*$. So $x \; R^{=} \; f(y)$, while $x \; R^{=} \; z$, so $z \circ_R f(y)$ and we are done.

Next we claim (ii) $(\forall y, z \in D) \; (y \equiv^{=}_{R} z \rightarrow f(y) = f(z))$. This is clear from Lemma 28(A), since if $y \equiv^{=}_{R} z$ then $y^* \subseteq z^*$ and vice-versa, so $y^* = z^*$. Putting together (i) and (ii) with Lemma 17, we have that f reduces $\langle D, R \rangle$ to $\langle C, Q \rangle$, hence we have part (A).

To get (B), given our hypotheses, we need only confirm that $Q =^{\tau} Q$ and $Q =^{\sigma} Q$. The former (which we have implicitly used) is trivial (cf. Lemma 18(A)), and we may get the latter by our proof of part (A), together with Lemma 20. (The fact that we did not need to assume strong supplementation for Q in our hypotheses may be seen as an aspect of Tarski's showing that Classical Mereology can be axiomatized as transitivity plus unique fusion existence; see [Hovda, 2009] for discussion.) $\dashv$

We elevate this fact to the status of Theorem because the reasoning in the proof we give contains the basic germ or pattern for characterizing a number of aspects of part-like relations under certain assumptions. Very roughly, if we imagine that there is some sub-domain of the universe that "looks like" a Classical Mereology under some (suitably tolerant) part-like relation, then, if everything in the rest of the universe can be traced back to this sub-domain along that part-like relation, then the universe is quasi-isomorphic with the sub-domain under that relation. This is born out in the following three Lemmas, the later Lemmas 36 and 39–42, and, especially, Theorem 6.

Lemma 30 *Suppose $\langle C, Q\rangle$ is a substructure of $\langle D, R\rangle$, $R =^{\tau} R$, $fld(R) \subseteq D$, and that D is $R^{=}$-grounded in C. Then $(\forall b \in fld(R))\ Fu_R(b, [R^{=}_{C}]\, b)$.*

Proof: It is trivial that $(\forall x \in [R^{=}_{C}]\, b)\ x\ R^{=}\ b$. Now suppose $z\ R^{=}\ b$; since $b \in fld(R) \subseteq D$, $z \in D$, and, by groundedness, there is a $w \in [R^{=}_{C}]\, z$; but then $w\ R^{=}\ b$, so we have $z \circ_R w$, and also $w \in [R^{=}_{C}]\, b$. $\dashv$

Lemma 31 *Suppose $R =^{\Sigma} R$, $fld(R) = D$, and that D is $R^{=}$-grounded in $C \subseteq D$. In general, let $Y^* = \bigcup\{[R_C]\, y : y \in Y\}$. Then: $(\forall b \in D)$ and $Y \subseteq D$:*

$$Fu_R(b, Y^*) \longleftrightarrow Fu_R(b, Y).$$

Proof: Deduce (recalling Lemma 28(B) and noting that $Y, Y^* \subseteq fld(R)$) that $Y \cong_{\circ_R} Y^*$ and apply Lemma 15. $\dashv$

Lemma 32 *Suppose $\langle C, Q\rangle$ is a substructure of $\langle D, R\rangle$, $fld(R) \subseteq D$, and $R =^{\Sigma} R$. Suppose also that $Q =^{\sigma} Q$, C is Q-fusion complete, and that D is $R^{=}$-grounded in C. Then $fld(R) = D$, $\langle D, R\rangle$ is quasi-isomorphic to $\langle C, Q\rangle$, and they reduce to a Classical Mereology.*

Proof: Q is transitive and so by Lemmas 25 and 27 we get that there is a c that reduces $\langle C, Q\rangle$ to a Classical Mereology $\langle C^{-}, Q^{-}\rangle$. Now, for any $x \in D$

there is a $y \in C$ with $y\ R^{=}\ x$. (To see that $fld(R) = D$, recall Lemma 22 and note that if $x = y$, then $x \in fld(Q) \subseteq fld(R)$, and if $x \neq y$, $x \in fld(R)$.) But then $c(y)\ R^{=}\ y$ while $c(y) \in C^{-}$; hence D is $R^{=}$-grounded in C^{-} and we may apply Theorem 4 to conclude that $\langle D, R\rangle$ reduces to $\langle C^{-}, Q^{-}\rangle$. $\dashv$

5 Merging domains

In this section we will be considering situations in which we have a domain of objects with a relation whose field is a subset of that domain, and a second, disjoint domain, with a relation whose field is a subset of the union of the two domains.

Notational convention for section 5: let D and E be disjoint non-empty sets, and let $F = D \cup E$; and let R be a relation on D (i.e. $fld(R) \subseteq D$), and S be a relation on F. We let P be $R \cup S$ (i.e., $\forall x \forall y (x\ P\ y \leftrightarrow (x\ R\ y \vee x\ S\ y)))$.

One application to bear in mind is this: think of D as the set of concrete (physical) objects or similar, and R as being the union of the primitive ways such things can be parts of one another. Then think of E as containing all the non-empty subsets of D, and take membership to be the primal part-hood relation for things in E, so that S would be the membership relation, (restricted to $D \cup E$). This application will nicely fit many of the general formal results we now work through. Alternatively, try thinking of E as the class of all (non-empty) sets that can be iteratively "built-up from" members of D (not including the empty set). Of course E is then not a set, but there are ways to adapt our results nonetheless – one simple way would be to stop the iteration at some point and let E be the set of sets generated up to that point.

5.1 Non-interference

On the condition that S never relates things within or into D, we get some nice results about induced part-like relations.

Lemma 33 *If* $\neg(\exists x \in F)(\exists y \in D)\, x\ S\ y$ *then:*
(0) $P \restriction D = R$
(1) $\neg(\exists x \in E)(\exists y \in D)\, x\ {}^{\tau}P\ y$
(2) $({}^{\tau}P) \restriction D = {}^{\tau}R$
(3) $(\circ_{{}^{\tau}P}) \restriction D = \circ_{{}^{\tau}R}$

(4) $(\forall x \in E)(\forall y \in D)\,(x \circ_{\tau P} y \to (\exists z \in D)(z\ {}^{\tau}P^{=}\ y \wedge z\ {}^{\tau}P\ x))$
(5) $({}^{\Sigma}P) \restriction fld(R) = {}^{\Sigma}R$

Proof: Clause (0) is easy. (1) is straightforward: any chain of P links that starts with a thing in E must stay in E, since each P link is either a R link or an S link, and R is confined to D, so to speak, while S never takes us into D, by hypothesis. (2) is clear for similar reasons: any chain of P links from $x \in D$ to $y \in D$ must stay within D and consist of R links. Hence $({}^{\tau}P) \restriction D \subseteq {}^{\tau}R$; the other direction is easy, since a chain of R links is a chain of P links, all within D. (3) is then easy. To get (4), if w witnesses $x \circ_{\tau P} y$, then, with $y \in D$, by (1), $w \in D$ as well, and witnesses the desired result. For (5): supposing $x \in fld(R)(= fld({}^{\tau}R))$ and $y \in fld(R)$ and $x\ {}^{\Sigma}P\ y$, we want to show that $\forall z(z \circ_{\tau R} x \to z \circ_{\tau R} y)$, so suppose $z \circ_{\tau R} x$. By (3), $z \circ_{\tau P} x$, and so by hypothesis $z \circ_{\tau P} y$, and another application of (3) gets us what we need. The other half of (5) is similar. $\dashv$

Note that with our hypotheses, even though we get $P \restriction D = R$, and clearly $fld(R) \subseteq fld(P)$, we are not guaranteed that $fld(P) \cap D \subseteq fld(R)$, since it could be that $x\ S\ y$ for $x \in D$ and $y \in E$ with $x \notin fld(R)$.

Some limited results obtain with a weaker stricture.

Lemma 34 *Suppose that* $E = E^{+} \cup E^{-}$ *and that* $(\forall x, y \in F)(x\ S\ y \to \neg[(x \in E^{+} \wedge y \in D) \vee (x \in E^{+} \wedge y \in E^{-}) \vee (x \in D \wedge y \in E^{-})]$*. Then:*
(0) $P \restriction D = R$
(1) $\neg(\exists x \in E^{+})(\exists y \in (D \cup E^{-}))\, x\ {}^{\tau}P\ y$
(2) $\neg(\exists x \in D)(\exists y \in E^{-})\, x\ {}^{\tau}P\ y$
(3) $({}^{\tau}P) \restriction D = {}^{\tau}R$

These are all straightforward to obtain. $\dashv$

On the condition that S never relates things within or out of D, we need a more stringent requirement to get similar results to Lemma 33, and even then there are limitations.

Lemma 35 *If* $\neg(\exists x \in D)(\exists y \in F)\, x\ S\ y$ *and* $(\forall x, y \in D)\,(x \circ_{\tau P} y \to x \circ_{\tau R} y)$*, then:*
(0) $P \restriction D = R$
(1) $\neg(\exists x \in D)(\exists y \in E)\, x\ {}^{\tau}P\ y$
(2) $({}^{\tau}P) \restriction D = {}^{\tau}R$
(3) $(\circ_{\tau P}) \restriction D = \circ_{\tau R}$
(4) $fld(P) \cap D = fld(R)$
(5) $({}^{\Sigma}P) \restriction D \subseteq {}^{\Sigma}R$

(0)–(2) are already covered by Lemma 34, (3) is immediate from our hypothesis. (4) also follows from our hypothesis, since, if $x \in D$ and $x \in fld(P)$, then $x \circ_{^\tau P} x$, hence $x \circ_{^\tau R} x$. (5) is then straightforward, but we cannot conclude that ${}^\Sigma R \subseteq ({}^\Sigma P) \restriction D$. ⊣

5.2 Articulation and fusion

Now we may show that if the domain E articulates or "comprehends," the domain D, in a certain sense, then there are fusions for arbitrary non-empty subsets of D, under both the ${}^\tau P$ relation and the ${}^\Sigma P$ relation, with respect to domain F.

There are various conditions on E that give fusion results. Here is simple one, that fits the idea of letting E be the set of non-empty subsets of D and S be membership (restricted to F) (and assuming that no subset of D is itself a member of D).

Lemma 36 *Suppose that for every non-empty $X \subseteq D$, there is at least one $y \in E$ such that (1) $(\forall x \in D)\,(x\ S\ y \leftrightarrow x \in X)$ and (2) $(\forall z \in E)\,(z\ S\ y \rightarrow z = y)$. Then:*
(A) for $X \subseteq D$ with $X \neq \emptyset$:
(A.1) $(\exists y \in E)\, Fu_{^\tau P}(y, X)$; and
(A.2) $(\exists y \in E)\, Fu_{^\Sigma P}(y, X)$.
(B) If E is ${}^\tau S^=$-grounded in D, then F is ${}^\Sigma P$-fusion complete.

Proof: For (A), fix non-empty $X \subseteq D$. Given y that witnesses the supposition, we may show that it also serves as both the desired fusions. For (A.1), the only non-obvious step is to show that, given $w \in F$ with $w\ {}^\tau P^=\ y$, $\exists x \in X\ x \circ_{^\tau P} w$. We may assume $w \neq y$ (since if $w = y$ then we are easily done). For any chain of P links that runs from w to y, we may consider the first link that runs to y; this link has to be either an R link or an S link; but by hypothesis it must be an S link running from some $x \in X$ to y; but then $w\ {}^\tau P^=\ x$ and hence $w \circ_{^\tau P} x$. For (A.2), invoke Lemma 14(A).

For (B), we first argue that $fld({}^\Sigma P) = F$: given $z \in F$, if $z \in E$, then, since E is ${}^\tau S^=$-grounded in D, get $x \in D$ with $x\ {}^\tau S\ z$, hence $z \in fld(S) \subseteq fld(P) = fld({}^\Sigma P)$; if $z \in D$, then by hypothesis get $y \in E$ with $x\ S\ y$ (consider $\{z\} \subseteq D$) hence again $z \in fld(S)$. We also may deduce that F is ${}^\Sigma P^=$-grounded in D. Now, given non-empty $Y \subseteq F$, letting $Y^* = \bigcup\{[{}^\Sigma P_D]y : y \in Y\}$, note that Y^* is non-empty, and use (A.2) to get a $b \in F$ such that $Fu_{\Sigma P}(b, Y^*)$. We may then invoke Lemma 31 to get that $Fu_{\Sigma P}(b, Y)$. $\dashv$

Definition 12 *(For any sets A, B, and relation Q): B Q-flows from A* $=_{def}$ *B is* ${}^{\tau}Q$*-grounded in A and* $\neg(\exists x \in A)(\exists y)\, y\ Q\ x$.

Note that, in general, if B Q-flows from A, then $B \cap A = \emptyset$ and $B \subseteq fld(Q)$.

Lemma 37 *If E S-flows from D, then* $\langle fld(R), {}^{\Sigma}R\rangle$ *is a substructure of* $\langle F, {}^{\Sigma}P\rangle$.

Proof: Check that the conditions of Lemma 33 part (5) apply, so $({}^{\Sigma}P) \restriction fld(R) = {}^{\Sigma}R$. $\dashv$

Lemma 38 *Suppose that E S-flows from D and* $y \in E$. *Then* $[{}^{\tau}S_D]\, y$ *is non-empty, and:*

(A) $(\forall z \in D)(z\ {}^{\tau}P\ y \leftrightarrow (\exists x \in [{}^{\tau}S_D]\, y)\ z\ {}^{\tau}P^{=}\ x)$
(B) $(\forall z \in E)(z\ {}^{\tau}P^{=}\ y \rightarrow [{}^{\tau}S_D]\, z \subseteq [{}^{\tau}S_D]\, y)$
(C) $Fu_{{}^{\tau}P}(y, [{}^{\tau}S_D]\, y)$
(D) $Fu_{\Sigma P}(y, [{}^{\tau}S_D]\, y)$

Proof: Note that for any $z_0 \in D$ such that $z_0\ {}^{\tau}P\ y$, there must be a chain $z_0\ P\ z_1$, $z_1\ P\ z_2$, $\ldots z_{n-1}\ P\ z_n$, where $z_n = y$ (and with the possibility that $n = 1$). For any such chain, there must be an $i \le n$ such that for $1 \le j \le i$, $z_j \in D$, and $z_{i+1} \in E$. Since R is confined to D and no member of E bears S to a member of D, we have that for $i + 1 \le j \le n$, $z_j \in E$, and, moreover $z_i\ S\ z_{i+1}$, $z_{i+1}\ S\ z_{i+2}$, and so forth; so $z_i\ {}^{\tau}S\ y$; hence $z_i \in [{}^{\tau}S_D]\, y$. This gives us (A). Lemma 28 gives us (B). Lemma 30 gives us (C), and (D) is then immediate from Lemma 14(A).

Lemma 39 *Suppose that E S-flows from D and that for every non-empty* $X \subseteq D$ *there is a* $y \in E$ *such that* $X = [{}^{\tau}S_D]\, y$. *Then F is* ${}^{\Sigma}P$*-fusion complete.*

Proof: We may deduce that $fld({}^{\Sigma}P) = F$ by an argument like that used for Lemma 36(B). We may deduce that F is ${}^{\Sigma}P$-grounded in D. And given a non-empty $Y \subseteq F$, letting $Y^* = \bigcup\{[{}^{\Sigma}P_D]y : y \in Y\}$, check that Y^* is non-empty; then our hypothesis gets us a $y \in E$ with $Y^* = [{}^{\tau}S_D]\, y$; may then invoke Lemmas 38(D) and 31 to get that $Fu_{\Sigma P}(y, Y)$. $\dashv$

Lemma 40 *Suppose that E S-flows from D and that for every non-empty $X \subseteq D$ there is an $X' \subseteq D$ with $X \cong_{\circ_{\Sigma P}} X'$ and $y \in E$ such that $X' = [{}^{\tau}S_D]\, y$. Then F is ${}^{\Sigma}P$-fusion complete.*

Proof: This follows easily from the previous Lemma together with Lemma 15. $\dashv$

This gives us a very general condition for the ${}^{\Sigma}P$-fusion completeness of F.

Lemma 41 *If D is ${}^{\Sigma}R$-fusion complete and E S-flows from D, then $\langle F, {}^{\Sigma}P\rangle$ is quasi-isomorphic to $\langle D, {}^{\Sigma}R\rangle$.*

Proof: Deduce that $fld({}^{\Sigma}R) = D$, recalling Lemma 22. Get that ${}^{\Sigma}P \restriction D = {}^{\Sigma}R$ from Lemma 33 part (5). Then invoke Lemmas 32 and 37. (For the applicability of Lemma 32, note that of course ${}^{\sigma}({}^{\Sigma}R) = {}^{\Sigma}R$.)$\dashv$

Lemma 42 *Suppose D is ${}^{\Sigma}R$-fusion complete, E S-flows from D, and for every non-empty $X \subseteq D$ there is a $y \in E$ such that $X = [{}^{\tau}S_D]\, y$. Then $\langle F, {}^{\Sigma}P\rangle$, $\langle E, ({}^{\Sigma}P) \restriction E\rangle$, and $\langle D, {}^{\Sigma}R\rangle$ are all quasi-isomorphic.*

Proof: Given the previous two Lemmas, it will suffice to show that $\langle F, {}^{\Sigma}R\rangle$ is quasi-isomorphic to $\langle E, ({}^{\Sigma}P) \restriction E\rangle$, and then appeal to the transitivity of quasi-isomorphism (for transitive relational structures), which is straightforward to deduce from Theorem 2. To show this, it is enough to note that $(\forall z \in F)(\exists y \in E)\ z \equiv^{=}_{\Sigma P} y$, and then apply Lemma 26. Given $z \in D$, then we note that by hypothesis there is a $y \in E$ with $[{}^{\tau}S_D]\, y = \{z\}$ and note that $Fu_{\Sigma P}(z, [x|x = z])$, and $Fu_{\Sigma P}(y, [x|x = z])$ (by Lemma 38(D)), and hence $y \equiv_{\Sigma P} z$ (recalling Lemma 14(E)).

5.3 Discussion

In one intuitive application of the above results, we let D be the set of concrete objects and R be the union of the primitive ways such things can be parts of one another. (Officially, we place no constraints on R, though there are some reasonable ones we could try). And we then let E be the set of all the non-empty subsets of D, and take membership to be the primal parthood relation for things in E, so that S would be the membership relation, (restricted to $D \cup E$). Then we have that E S-flows from D, and we can apply our recent Lemmas to deduce that $\langle F, {}^{\Sigma}P\rangle$ is quasi-isomorphic to a Classical Mereology, among other things.

Moreover, suppose we iterate the basic process that took us from $\langle D, R\rangle$ to $\langle F, {}^{\Sigma}P\rangle$: we take E^2 to be the set of non-empty subsets of F, and let S^2 be the membership relation restricted to $F^2 = F \cup E^2$. And we set $P^2 = {}^{\Sigma}P \cup S^2$. Then E^2 S^2-flows from F, and we may apply our Lemmas to get that $\langle F^2, {}^{\Sigma}P^2\rangle$ is quasi-isomorphic to $\langle F, {}^{\Sigma}P\rangle$. Hence, if we differentiate structures only "up to quasi-isomorphism," he find that iterating this process does not alter structure. In effect, iterating just places more nodes within clusters of "equivalent" items: e.g., if $z \in E^2$, then there is an $x \in F$ such that $x \equiv_{{}^{\Sigma}P^2} z$ and for all $y \in F$ $y \equiv_{{}^{\Sigma}P^2} z \leftrightarrow y \equiv_{{}^{\Sigma}P} x$.

The basic phenomena generalize. To illustrate, let $E^1 = E = \{w : w \neq \emptyset \wedge w \in \mathcal{P}(D)\}$, $E^2 = \{w : w \neq \emptyset \wedge w \in \mathcal{P}(D \cup E^1)\}$ and so forth, and let E^ω be $\{x : x \in E^n$ for some finite $n\}$. Let G be any subset of E^ω, let S^G be the membership relation restricted to $H = D \cup G$, and let P^G be $R \cup S^G$. Then H is ${}^{\tau}P^G$-grounded in D, and G ${}^{\Sigma}P^G$-flows from D. We will show in a moment (Theorem 6) that $\langle G, {}^{\Sigma}({}^{\Sigma}P \cup S^G)\rangle$ is quasi-isomorphic to a substructure of $\langle F, {}^{\Sigma}P\rangle$. We note that if G includes "enough" sets from E^ω, then the conditions of Lemma 39 apply, and (as we will show), $\langle G, {}^{\Sigma}({}^{\Sigma}P \cup S^G)\rangle$ is quasi-isomorphic to $\langle F, {}^{\Sigma}P\rangle$.

And there is nothing peculiar about *sets*, as opposed to other objects that "flow from" our base. Roughly: any additional objects under a relation on which they flow from our base D are structurally bound either to "join in" to existing equivalence clusters, or to fill in "gaps" in places where there is a cluster in the structure that arises when enough objects are added to yield fusion-completeness.

6 Completions and general structure

We now turn to making this last remark more precise.

Definition 13 *Say that* $\langle F, P\rangle$ *is a completion of* $\langle D, R\rangle$ *when*
(i) there are relations R° *and* P°*, with* $D = fld(R^\circ)$*,* $F = fld(P^\circ)$*,* $R = {}^{\Sigma}R^\circ$ *and* $P = {}^{\Sigma}P^\circ$*; and*
(ii) $\langle D, R\rangle$ *is a substructure of* $\langle F, P\rangle$*; and*
(iii) F *is* P*-fusion complete; and*
(iv) F *is* P*-grounded in* D*.*

Theorem 5 *For any relation R°, there is a completion of $\langle fld(R^\circ), {}^\Sigma R^\circ\rangle$.*

Proof: Our result is basically evident from our discussion thus far. Officially, we are assuming here that R° is a set of ordered pairs. Given that $D = fld(R^\circ)$ is a set, we can consider letting $E = \{X : X \neq \emptyset \wedge X \subseteq D\}$, $F = D \cup E$, $P^\circ =$ the union of R° with membership restricted to F, and $P =^\Sigma P^\circ$. Assuming that $E \cap D = \emptyset$, we have seen (cf. Lemmas 36 and 37) that then $\langle F, P\rangle$ is a completion of $\langle D, {}^\Sigma R^\circ\rangle$. If some subsets of D are members of D, then we will need some other construction. But we should still be able to find some set of objects E' and relation S' to play the roles played by E and S in Lemma 36 or 39 or 42. If we are working with ZFC, then we can use sets whose rank is higher than that of D. In other set theories, there should be similar resources available, unless R° is "too big" to be a set anyway. $\dashv$

Theorem 6 *For any relation R°, with $D = fld(R^\circ)$ and $R =^\Sigma R^\circ$, if $\langle F, P\rangle$ is a completion of $\langle D, R\rangle$ then:*

for any $\langle G, Q\rangle$ of which $\langle D, R\rangle$ is a substructure, where $Q =^\Sigma Q^\circ$ (with $G = fld(Q^\circ)$) and where G is ${}^\tau Q^{\circ=}$-grounded in D:

(A) $\langle G, Q\rangle$ is quasi-isomorphic to a substructure of $\langle F, P\rangle$; and

(B) if G is Q-fusion complete, then $\langle G, Q\rangle$ is quasi-isomorphic to $\langle F, P\rangle$.

Proof: By Theorem 3, we can get a c that reduces $\langle F, P\rangle$ to a Classical Mereology $\langle F^-, P^-\rangle$. Let d reduce $\langle G, Q\rangle$ to $\langle G^-, Q^-\rangle$. Our main goal is to find isomorphism h from $\langle G^-, Q^-\rangle$ to some substructure of $\langle F^-, P^-\rangle$. Given $y \in G^-$, since G is ${}^\tau Q^{\circ=}$-grounded in D, G is $Q^=$-grounded in D, (and $y \in fld(Q)$), and we may apply Lemma 30 to get that $Fu_Q(y, [Q_D]\, y)$. (Note that for any $y \in G, [Q_D]y = [Q^={}_D]y$, since Q is reflexive on $G = fld(Q)$.) Now, since $\langle F, P\rangle$ is P-fusion complete, we get a $y' \in F$ with $Fu_P(y', [Q_D]\, y)$, and we may set $h(y) = c(y')$. This is well-defined, since for any $y'' \in F$ with $Fu_P(y'', [Q_D]\, y)$, $y' \equiv_P y''$, hence $c(y') = c(y'')$.

Now for any $y, z \in G^-$, we get $y', z' \in F$ with $Fu_P(y', [Q_D]\, y)$ and $Fu_P(z', [Q_D]\, z)$ and $h(y) = c(y')$ and $h(z) = c(z')$. We claim that:

$$y\ Q^-\ z \longleftrightarrow h(y)\ P^-\ h(z).$$

This follows from the correctness of these four biconditionals:

$$
\begin{aligned}
y\ Q^-\ z &\longleftrightarrow_{(1)} y\ Q\ z \\
&\longleftrightarrow_{(2)} (\forall x \in [Q_D]y)(\exists w \in [Q_D]z)\ x \circ_Q w \\
&\longleftrightarrow_{(3)} y'\ P\ z' \\
&\longleftrightarrow_{(4)} c(y')\ P^-\ c(z').
\end{aligned}
$$

Biconditionals (1) and (4) are covered by Lemma 18. (2) is covered by Lemma 29. For the left-to-right direction of (3), bearing in mind both (B.1) and (B.3) of Lemma 29, we have that $[Q_D]\, y \subseteq [Q_D]\, z$, and so if $a\ P\ y'$, then there is $x \in [Q_D]z$ with $a \circ_P x$ and hence $a \circ_P z'$; now appeal to Lemma 10. For the right-to-left direction: given $x \in [Q_D]\, y$, we have $x\ P\ y'\ P\ z'$. Thus there is a $w \in [Q_D]\, z$ such that $x \circ_P w$. Bearing in mind that F is P-grounded in D, by Lemma 28(B), we get a $b \in D$ that witnesses this overlap: $b\ P\ x$ and $b\ P\ w$. But then $b\ R\ x$ and $b\ R\ w$, and hence $b\ Q\ x$ and $b\ Q\ w$. Thus $x \circ_Q w$, and we are done.

Now, since P^- is reflexive, if $h(y) = h(z)$, $h(y) \equiv_{P^-} h(z)$, so $y \equiv_{Q^-} z$. Since Q^- is anti-symmetric, we then have $y = z$. Hence h is one-to-one.

Letting E be the range of h, we may conclude that $h : G^- \to E$ is an isomorphism from $\langle G^-, Q^- \rangle$ to $\langle E, P^- \restriction E \rangle$, which is a substructure of $\langle F^-, P^- \rangle$, hence of $\langle F, P \rangle$. This completes the argument for part (A).

For part (B), let $x \in F^-$. $[P_D]\, x$ is non-empty; by our additional hypothesis, there is a $y \in G$ with $Fu_Q(y, [P_D]\, x)$. As we noted above, we know $Fu_Q(y, [Q_D]y)$, so, recalling Lemma 15, we have $[P_D]x \cong_{\circ_Q} [Q_D]y$. Now we argue that $[P_D]x \cong_{\circ_P} [Q_D]y$: if $b \in [P_D]x$ and $a \circ_P b$, then (Lemma 28) get $w \in D$ with $w\ P^=\ a$ and $w\ P^=\ b$. But since b also $\in D$, we have $w\ R^=\ b$ and $w\ Q^=\ b$. So $w \circ_Q b$, so get $e \in [Q_D]y$ with $w \circ_Q e$. Deduce $w \circ_P e$, hence $a \circ_P e$. Parallel reasoning runs the other way to yield $[P_D]x \cong_{\circ_P} [Q_D]y$. But we also have $Fu_P(h(d(y)), [Q_D]\, y)$; appealing again to Lemma 15, we get $Fu_P(h(d(y)), [P_D]x)$. But also $Fu_P(x, [P_D]\, x)$, hence $x \equiv_P h(d(y))$. But x and $h(d(y))$ are both in the range of c, so $x = c(x) = c(h(d(y)) = h(d(y))$. So x is in the range of h, and thus $E = F^-$. $\dashv$

6.1 Discussion

Imagine again that we start with some domain D and some relation R° on D (with $fld(R^\circ) = D$ that we take to capture some or all primitive parthood connections among things in D (including identity on D; recall our discussion of reflexivity above). We consider ${}^{\tau}R^\circ$ and ${}^{\Sigma}R^\circ$ to be derivative parthood relations on D. Now we note that there may be "new" (abstract, perhaps) objects E that could be "built out of" the things in D, and we imagine that the primitive parthood relation S° that connects these new things to D, and might also connect them to one another, "flows from" D in basically the sense made precise above. We might think of S° as marking exactly the genesis of a new thing: $x\ S^\circ\ y$ just when y was immediately generated by, or built from, (possibly among other things) x. Then ${}^\tau S^\circ$ traces the ancestry of a new thing and is itself a parthood relation. When we take view of the domain $F = E \cup D$, we see that $P^\circ = R^\circ \cup S^\circ$ represents the sum of the primitive parthood relations on this domain, and, accordingly ${}^\tau P^\circ$ and ${}^\Sigma P^\circ$ represent derivative parthood relations. Within the domain D, these relations are no different from ${}^\tau R^\circ$ and ${}^\Sigma R^\circ$ (cf. Lemma 29).

Let us say that two quasi-isomorphic structures have the same q-type, and that one q-type is a sub-type of another if any structure with the first q-type is quasi-isomorphic to a sub-structure of a structure with the second q-type. Then Theorem 6 tells us that there is a unique q-type (fixed by the q-type of the initial $\langle D, R\rangle$) that the q-type of any such $\langle F, {}^\Sigma P^\circ\rangle$ must be a sub-type of: this is the q-type of a completion of $\langle D, R\rangle$.

For another perspective, consider the addition of a single new object y to a given $\langle D, R^\circ\rangle$, letting $R = {}^\Sigma R^\circ$. Think of y as being generated by some thing or things in D, and let S° mark this generation, so that $[S^\circ_D]y$ is precisely the set of things that generated y. Now letting $F = D \cup \{y\}$ and $P^\circ = R^\circ \cup S^\circ$ and $P = {}^\Sigma P$, there are only two possibilities for the q-type of $\langle F, P\rangle$, *vis a vis* $\langle D, R\rangle$. We know (Lemmas 37 and 30) that $Fu_P([P_D]\,y)$. The two possibilities are whether or not there is already $z \in D$ such that $Fu_R(z, [P_D]\,y)$. If there is such a z, then (Lemma 14(D)) $z \equiv_P y$, and it is easy to see that $\langle F, P\rangle$ is quasi-isomorphic to $\langle D, R\rangle$, so that the new structure has exactly the same q-type as the old one. If not, then there is no $z \in D$ such that $z \equiv_P y$. This is because if there were such a z, we would have $Fu_P(z, [P_D]\,y)$, hence (cf. Lemma 28(B)) $Fu_R(z, [P_D]\,y)$. In this case, $\langle F, P\rangle$ is not quasi-isomorphic to its substructure $\langle D, R\rangle$, and y occupies a new position, so to speak. The q-type of $\langle D, R\rangle$ is then a proper sub-type of the q-type of $\langle F, P\rangle$. Thus we can think of $\langle D, R\rangle$ as having "gaps" precisely where it lacks R-fusions for various subsets of D.

We might imagine now adding a new thing y' to F in just the manner in which y was added to D, and so forth. (One can check that each of the resulting structures flows from $\langle D, R\rangle$, and it does not really matter whether we add objects and generate structures cumulatively, in this manner, or all at once as in our previous discussion.) At each step in the cumulative process, when a new object is added, it may fill in a "gap" in the pattern of the previous structure. If we reach a stage without gaps, then any further additions will generate nothing new *modulo* quasi-isomorphism. We will have reached a fixed point: all further additions will have the same q-type.

Again, we were thinking of the S° relations as being the primitive ways in which the new things have parts. It is the less discerning derivative relations, especially the ones in the role of P, for which we were able to give a strong structural characterization.

6.2 Expansion without groundedness

We have been focusing on what happens when we "extend" a given domain (with part-like relations on it) to a larger domain, where the new objects are born new primitive part-like relations *from* the old ones. Our main result was roughly that, relative to one particular sort of tolerant parthood relation, there is an aspect in which there is a unique pattern that can result, fixed by the pattern in the given seed. But what if some new objects are not connected to the old ones in such a manner? We made some very brief points about such situations in Lemmas 34 and 35, but did not explore very far.

There are two very different cases: (1) a new object might simply be disconnected from the old domain, with no parthood relations running from it to the old domain, nor to it, from the old domain; or (2) a new object might be a primitive part of an old object. Roughly speaking, in case (1) the new object is guaranteed not to "disturb" the old domain and its parthood relations, in a fairly strong sense: we can merge the new object and its relation into the old domain and relation, take the transitive closure and overlap-inclusion of the merged relations, and the restrictions of these relations will be the corresponding old ones. But in case (2), there is no such guarantee: for example, when we merge, old objects that did not overlap under the old relation might overlap under the new one, and hence the restriction, to the old domain, of the overlap-inclusion of the transitive closure of the merged relation might not be identical with its old counterpart. Now there is some reason to insist on metaphysical grounds that non-concrete objects cannot bear genuine parthood relations to concrete ones; this is natural when we think of the non-concrete ones as either independent of (think "Platonic") or generated by (think "Aristotelean") concrete ones. But there may be some reason to think otherwise; (see section X of [Fine, 2010] for some motivations). There is not room for further discussion here, but we may hope that some of the analytical tools developed above could be of use in exploring the formal patterns that can emerge, given some reasonable constraints.

References

[Fine, 2010] Fine, K. (2010). Towards a theory of part. *Journal of Philosophy*, 107(11):559–589.

[Hovda, 2009] Hovda, P. (2009). What is classical mereology? *Journal of Philosophical Logic*, 38(1):55–82.

[Hovda, 2016] Hovda, P. (2016). Parthood-like relations: Closure principles and connections to some axioms of classical mereology. *Philosophical Perspectives*, 30(1):183–197.

[Simons, 1987] Simons, P. (1987). *Parts: A Study in Ontology*. Oxford University Press.

[Varzi, 2016] Varzi, A. (2016). Mereology. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, winter 2016 edition.