TRIGONOMETRY

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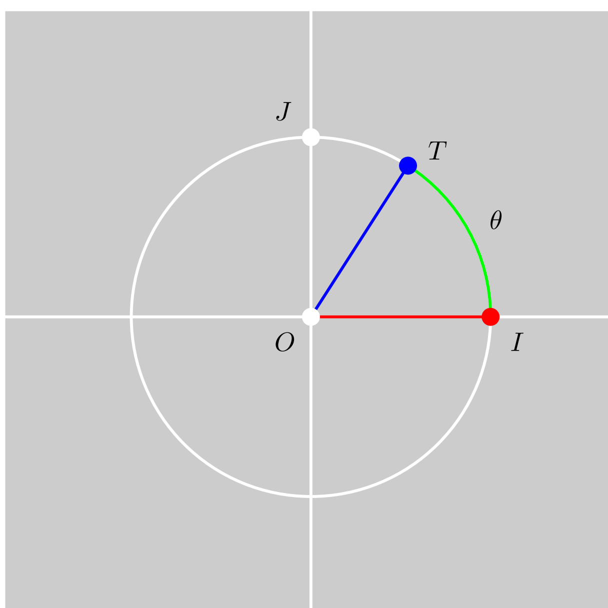
Introduction

01° In our discussions of Light, we encounter, repeatedly, the phenomenon of refraction. The governing principle for this phenomenon is the Law of Snell. The law depends upon the elements of Trigonometry. Just the simplest of these elements are sufficient for our purposes.

Angular Measure

02° We introduce Cartesian Coordinates for the Euclidean Plane. Of course, we set the same units of length for the two axes. In turn, we introduce a circle Γ in the plane for which the center O is the origin of the coordinates and the radius is one unit. Finally, we draw an angle A in the plane, carefully, in "standard position." That is, we set the vertex of A at O, the initial side along the first coordinate axis in the positive direction, and the terminal side wherever it falls. For neatness, we trim the sides to a length of one unit.

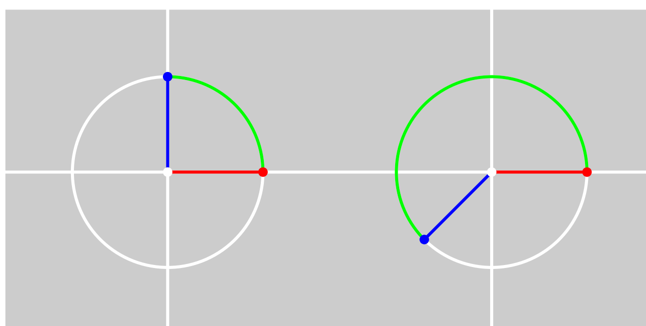


$A, \quad \theta$

03° The initial points of the initial and terminal sides of A both coincide with $O = (0, 0)$. The terminal point of the initial side is determined. It is $I = (1, 0)$. The terminal point of the terminal side is whatever it proves to be, say $T = (x, y)$. By ccw transition, the points I and T define an "arc" of the circle Γ, let it be θ.

04° Now we require a precise measure of the angle A. To that end, we might as well identify A with θ. We prefer θ.

05° Since the radius of Γ is 1, the circumference of Γ is 2π. Naturally, we think to measure θ in proportion to 2π. We refer to this measure as the "radian" measure of θ, let it be $\rho(\theta)$. For example, if θ is one quarter of Γ then $\rho(\theta) = \pi/2$ radians. If θ is five eights of Γ then $\rho(\theta) = 5\pi/4$ radians. If θ equals Γ itself then $\rho(\theta) = 2\pi$ radians.



Radians

06• Draw the angle for which the radian measure is 1. (See article 08°.)

07° But we ought to leap backward in time to note that there is a "popular" definition of angular measure, in terms of "degrees." It stems from the ancient practice of breaking the year into 360 days, subject to continual correction. For our purposes, we pretend that the circumference of Γ is not 2π radians but 360 degrees. We think to measure θ in proportion to 360. We refer to this measure as the "degree" measure of θ, let it be $\delta(\theta)$. For example, if θ is one quarter of Γ then $\delta(\theta) = 90$ degrees. If θ is five eights of Γ then $\delta(\theta) = 225$ degrees. If θ equals Γ itself then $\delta(\theta) = 360$ degrees. Often, we write $n°$ instead of "n degrees."

08° Conversion between the radian and degree measures of an angle is easy:

$$\delta(\theta) = \frac{360}{2\pi}\rho(\theta), \qquad \rho(\theta) = \frac{2\pi}{360}\delta(\theta)$$

Cosine and Sine

09° Now we can define the celebrated functions Cosine and Sine for a given angle A. To this end, we identify A, θ, and $\rho(\theta)$. We prefer θ. The values of the functions are simply the first and second coordinates of the point T:

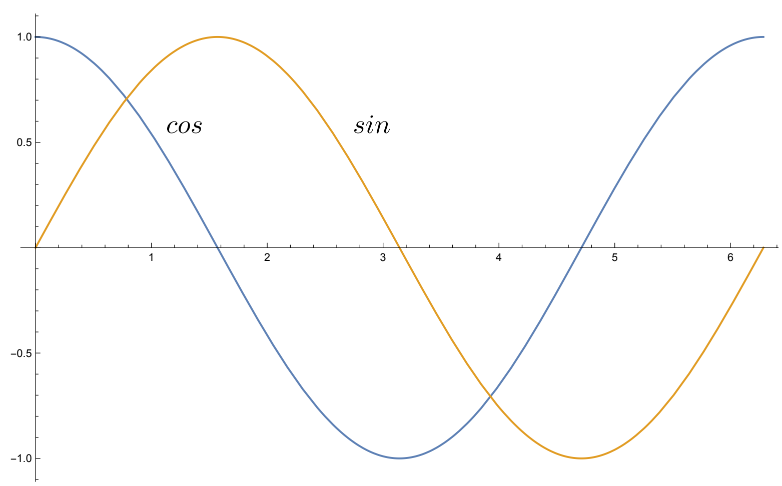
$$T = (x, y) \qquad \text{implies} \qquad cos(\theta) = x \quad \text{and} \quad sin(\theta) = y$$

See the diagram in article 2°.

10° Since $x^2 + y^2 = 1$ we find right away that:

$$cos^2(\theta) + sin^2(\theta) = 1$$

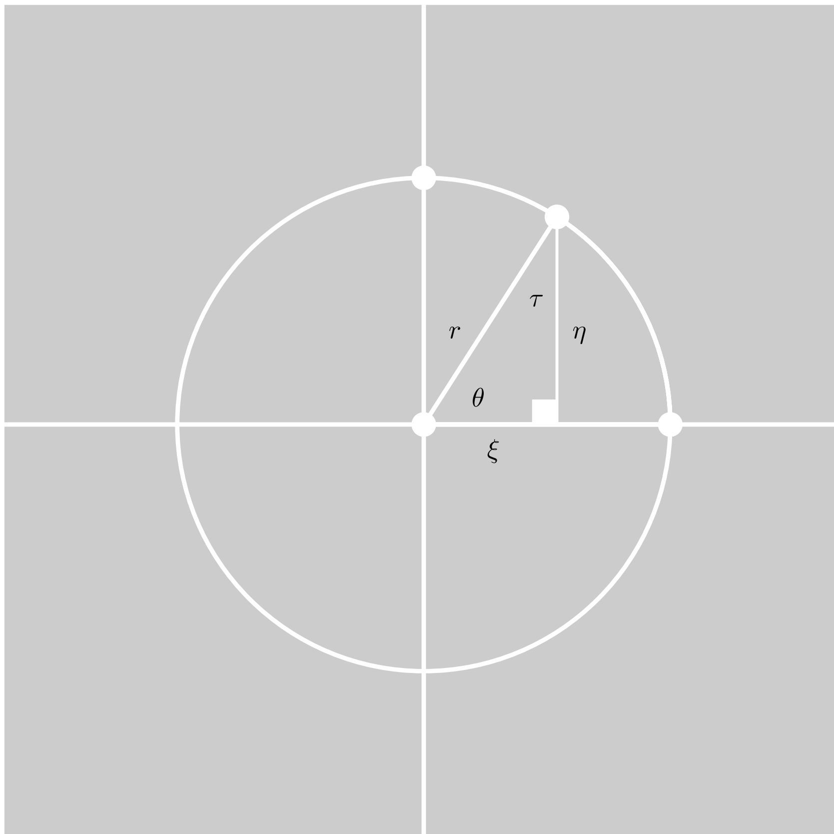
For graphs of these functions, see the following figure.



$cos, \quad sin$

11• Identify the values of θ for which $cos(\theta)$ and $sin(\theta)$ equal 0, and the values of θ at which $cos(\theta)$ and $sin(\theta)$ reach minimum value -1 or maximum value $+1$.

12° For the important case in which $0 \leq \rho(\theta) \leq \pi/2$, we can translate our description of the Cosine and Sine functions into a more familiar form, based upon a right triangle. The diagram in article 2° shows the way.



Old Familiar

13° We recover the classical definitions of Cosine and Sine for right triangles:

$$cos(\theta) = \frac{\xi}{r}\ , \quad sin(\theta) = \frac{\eta}{r}$$

In this setting, we have changed the radius of the circle Γ to r, so that the length of the hypotenuse of the triangle is not 1 but r.

14° One should note that $\theta + \tau = \pi/2$ and that, in such a case:

$$cos(\tau) = sin(\theta) \quad \text{and} \quad sin(\tau) = cos(\theta)$$

These facts lead directly to Brewster's Angle.