THE CARTESIAN RAINBOW

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01° In 1637a, René Descartes published one of the celebrated works in the history of thought: **Discours de la Méthod**. In the preface to this work, Descartes set forth the central precepts of his scientific method, by which one would "acquire knowledge and avoid error." In the work itself, he presented three substantial discourses, which served as grand instances of successful application of his method. However, in course of time, the preface came to be viewed (and published) in its own right as a fundamental exposition of the scientific method.

The three discourses by Descartes were devoted to Geometry, Dioptrics, and Meteorology. In the first, Descartes initiated the study of geometry by arithmetic methods, that is, by means of coordinate systems. In the second, he described a quantitatively precise expression for the relation between the incident and refracted rays in context of refraction of light, the relation now known as the Law of Snell. Finally, in the third, he applied the Law of Snell to develop a compelling explanation of the Rainbow.

The Problem

02° In his **Meteorologica** (c0340b), Aristotle presented the rainbow as a problem to be solved. He required a description of:

(•) the agents of formation of the rainbow

and he required explanations of:

(•) its shape
(•) its size
(•) and its colors

Ab initio, Aristotle identified the agents of formation of the rainbow as the Sun, a rain shower, and the eye of an observer. He declared its shape to be a circular arc. These contributions have proved durable. The rest of his ideas, however, have proved misleading.

03° In his **Magnum Opus**, delivered to Pope Clement IV in 1268a, Roger Bacon reported his measurement of the angle of elevation of the peak of the rainbow at sunset: roughly 42°. This number serves as a measure of the size of the bow.

04° The concentric arcs of color in the rainbow, descending subtly through the visual spectrum:

$$red, orange, yellow, green, blue/indigo/violet$$

comprise the primary mystery. For more than two thousand years, efforts to explain the colors have developed, step for step, with efforts to explain the nature of Light itself.

Descartes' Diagrams

05° Let us summarize Descartes' explanation of the shape and size of the rainbow. In the first of the following two diagrams (Figure 1), one finds the Sun setting in the west, rain falling in the east, and an (astonished) observer taking note of the rainbow formed in the sky by the interaction of rays of light from the Sun and droplets of water in the shower.

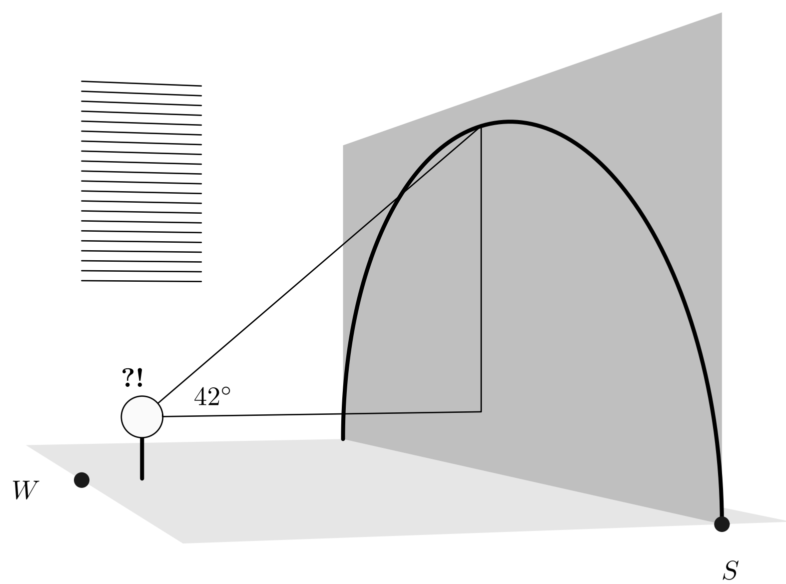

Figure 1: Observation

06° In the second diagram (Figure 2), one finds a particular ray of light and a particular raindrop magnified for inspection. The parameter y measures the elevation of the particular ray above the indicated axis of the raindrop. We have set the radius of the raindrop at one unit. In reality, the radius is roughly one millimeter. The stream of Particles composing the ray will meet the raindrop at point A, some being reflected but some being refracted into the body of the drop. Those particles which enter the drop at point A will meet the opposite surface at point B, some being refracted into the exterior but some being reflected. The particles which are reflected at point B will again meet the surface of the raindrop at point C, some being again reflected but some being refracted into the exterior. The particles which leave the raindrop at point C will have followed the path drawn in the diagram. Employing the Law of Snell, Descartes calculated the angle δ of deviation of the incident ray as a function of the parameter y:

$$(\star) \qquad \delta = \pi + 2\iota - 4\rho$$

where ι and ρ are the angles of Incidence and Refraction, respectively, as indicated in the diagram. Of course, ι and ρ are determined by y. See article 9°.

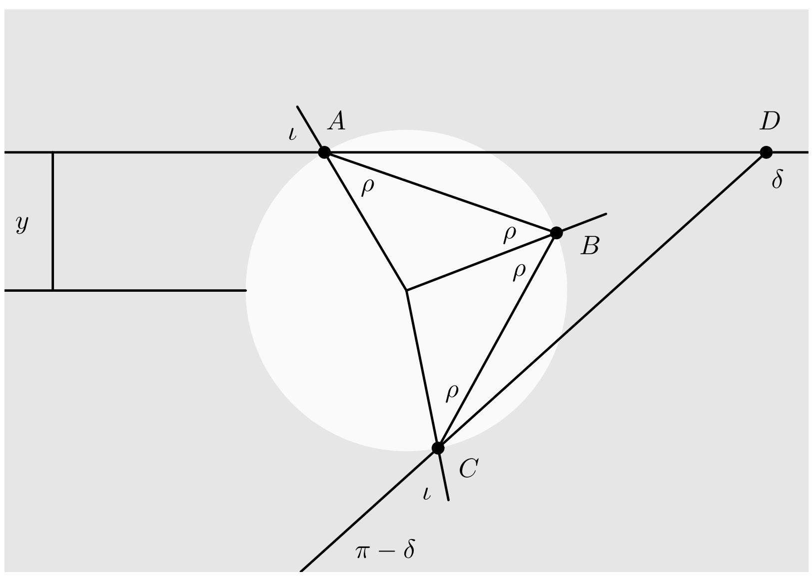

Figure 2: Deviation

Derivation of the Deviation Angle δ

07° The incident ray of light marked by the parameter y $(0 < y < 1)$ changes direction three times: at point A, at point B, and at point C. At point A, it turns clockwise through an angle of $\iota - \rho$; at point B, clockwise through an angle of $\pi - 2\rho$; and at point C, clockwise through an angle of $\iota - \rho$. Hence, the total angle δ of deviation of the incident ray is $\pi + 2\iota - 4\rho$.

The Calculations

08° In his famous tables of trigonometric functions (1612a), Bartholomeus Pitiscus recorded the values of the sine, tangent, and reciprocal cosine functions accurate to seven significant figures in steps of one sixth of one sixtieth of a degree. With immense patience, Descartes applied the tables to calculate approximate values of δ corresponding to the following values of y:

$$0.0,\ 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0$$

Intrigued by the emerging pattern, he then calculated approximate values of δ corresponding to the following values of y:

$$0.80,\ 0.81,\ 0.82,\ 0.83,\ 0.84,\ 0.85,\ 0.86,\ 0.87,\ 0.88,\ 0.89,\ 0.90$$

Assembling the numbers in a graph, he found compelling evidence that one special value of y, roughly 0.86, yielded a corresponding deviation angle δ of minimum value:

$$\frac{180^\circ}{\pi}\delta \approx 138^\circ$$

Let us denote those values of y and δ by $\bar{y}$ and $\bar{\delta}$ and let us refer to the ray with parameter $\bar{y}$ as the Cartesian Ray. Clearly, the cartesian ray will reach the eye of the observer at an angle of elevation:

$$180^\circ - \frac{180^\circ}{\pi}\bar{\delta} \approx 42^\circ$$

the angle of Bacon.

The Basic Graph

09° Let us apply the Calculus to analyze relation $(\star)$. Let Δ be the *deviation* function having domain $(0,1)$, defined as follows:

$$\delta \equiv \Delta(y) := \pi + 2\iota - 4\rho$$

where y is any number in $(0,1)$. Of course, ι and ρ are determined by y. In fact, by Figure 2, $y = sin(\iota)$ and $\iota = arcsin(y)$. By the Law of Snell:

$$\boxed{sin(\iota) = \nu\, sin(\rho)}$$

Hence:

$$y = \nu sin(\rho), \quad \rho = arcsin(\frac{1}{\nu}y)$$

Now we can present Δ explicitly as a function of y:

$$\delta \equiv \Delta(y) := \pi + 2\,arcsin(y) - 4\,arcsin(\frac{1}{\nu}y)$$

where y is any number in $(0,1)$.

10° Descartes adopted the following value for the air/water Index of Refraction ν:

$$\nu = \frac{4}{3}$$

11° By differentiation, we find that:

$$\frac{d\delta}{dy} = 0 + 2\frac{d\iota}{dy} - 4\frac{d\rho}{dy} = \frac{2}{\sqrt{1-y^2}} - \frac{4}{\sqrt{\nu^2 - y^2}}$$

By simple computation, we find that:

$$\frac{d\delta}{dy} < 0 \quad \text{iff} \quad y < \sqrt{\frac{4-\nu^2}{3}}$$
$$\frac{d\delta}{dy} = 0 \quad \text{iff} \quad y = \sqrt{\frac{4-\nu^2}{3}}$$
$$\frac{d\delta}{dy} > 0 \quad \text{iff} \quad y > \sqrt{\frac{4-\nu^2}{3}}$$

Clearly, the graph of Δ must take the form displayed in Figure 3. Moreover:

$$\bar{y} = \sqrt{\frac{4-\nu^2}{3}}$$

and:

$$\bar{\delta} = \pi + 2\,arcsin(\bar{y}) - 4\,arcsin(\frac{1}{\nu}\bar{y})$$

For the value $\nu = 4/3$, we find that:

$$\bar{y} = 0.8607$$

and:

$$180° - \frac{180°}{\pi}\bar{\delta} = 180° - \frac{180°}{\pi}2.4080 = 180° - 138.0° = 42.0°$$

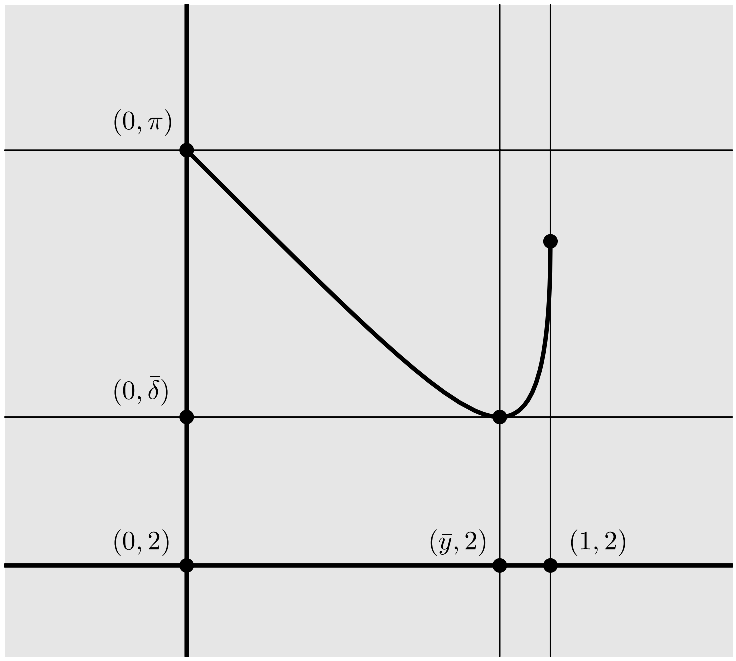

Figure 3: The Cartesian Graph: $\delta = \Delta(y)$

12° The foregoing analysis makes sense only if $1 < \nu < 2$.

13° Let us introduce the *elevation* function H, having domain $(1,2)$:

$$\epsilon \equiv H(\nu) := \pi - \bar{\delta} = 4\,arcsin(\frac{1}{\nu}\sqrt{\frac{4-\nu^2}{3}}) - 2\,arcsin(\sqrt{\frac{4-\nu^2}{3}})$$

where ν is any number in $(1,2)$. With diligence, one can show that:

$$\frac{d\epsilon}{d\nu} = \cdots\cdots\cdots = -\frac{2}{\nu}\sqrt{\frac{4-\nu^2}{\nu^2-1}}$$

Hence, H is strictly decreasing. See the following article 20°.

Interpretation: Its Size

14° The coincidence between Descartes' calculation and Bacon's measurement is, of course, striking. However, it does not by itself constitute an **explanation** of the size of the rainbow. Scientific explanation requires more than a coincidence between construction and measurement. It requires that the coincidence itself be subject to rational interpretation. The process of explaining natural phenomena is inherently regressive, terminating only when it reaches a primary layer of uncontested assent.

15° But Descartes pressed his discovery to a deeper level. He called attention to the **significance** of the minimum value of a function. Since the cartesian ray yields a deviation angle of minimum value, the light rays nearby to that ray will emerge from the raindrop *closely packed*. They will create for the eye of the observer the impression of a bright spot in the sky at an angular elevation of 42°. In contrast, the light rays far from the cartesian ray will emerge more or less evenly spaced and, in comparison with the Cartesian Pack, will create for the eye of the observer impressions substantially less bright. The following Ray Diagram (Figure 4) makes everything clear.

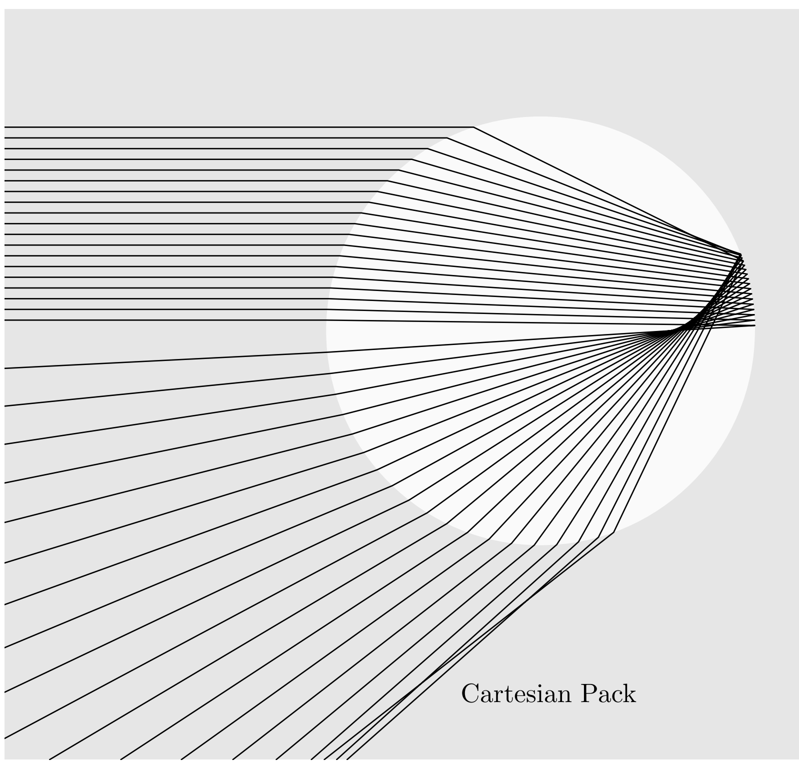

Figure 4: Ray Diagram

The Cinematic Effect

16° Of course, raindrops fall. While the bright spot seems to hang in the sky at an angular elevation of 42°, the raindrops creating it give way, moment by moment, to those above. Moreover, raindrops fall rapidly. The bright spot seems to hang continuously. It does not flicker.

Interpretation: Its Shape

17° Descartes' construction is symmetric about the line issuing from the eye of the observer, parallel to the line of the horizon. Accordingly, any raindrop for which the angle between the eye-raindrop line and the eye-horizon line is 42° will contribute to the impression of the rainbow for the observer. Technically, then, the rainbow consists of a circular cone of *directions*, with vertex at the eye of the observer and with angle of aperture equal to 42°. Hence, Descartes' construction explains not only the size but also the circular shape of the rainbow.

Jubilation

18° Descartes attempted but failed to explain the distribution of colors in the rainbow.

19° While the theory of Descartes would in due course prove to be only the first step in a complex sequence of refinements, continuing to the present day, one can hardly help but share in his jubilation:

> *"Those who have understood all which has been said in the treatise will no longer see anything in the clouds in the future for which they will not easily understand the cause."* (**Les Météors**)

Its Colors

20° In 1704a, Isaac Newton published his treatise: **Optiks**. In this work, Newton presented his theory of color and his application of that theory to numerous observations of natural bodies. In particular, in Proposition 9, Problem 4 of Book 1, Part 2, he set the following problem:

> *"By the discovered properties of light, to explain the colours of the rainbow."*

To solve the problem, Newton applied the theory of Descartes but he introduced a new feature: the parameter ν (the index of refraction for air/water) varied for various colors of visible light, being smallest for red light and largest for blue:

$$\nu_r = 1.331$$
$$\nu_b = 1.343$$

As a result, the angular elevation of the cartesian pack varied for various colors:

$$\frac{180.00°}{\pi}\epsilon_r = 42.37°$$
$$\frac{180.00°}{\pi}\epsilon_b = 40.65°$$

where $\epsilon_r := H(\nu_r)$ and $\epsilon_b := H(\nu_b)$. Newton's theory entailed that the vertical span of the rainbow should be:

$$42.37° - 40.65° = 1.72°$$

which proved to be in rough agreement with observation.

21° One applies the term Dispersion to refer to optical phenomena which depend specifically upon the color (that is, the Frequency) of light. Thus, one may say that the distribution of colors in the rainbow is an effect of dispersion. However, one may rightly ask whether such a statement explains anything at all. The fact of dispersion appears as an empirical irreducible. Even under the sophisticated theory of Electricity and Magnetism perfected by James C. Maxwell in the Nineteenth Century, the effects of dispersion are traceable to the empirically determined parameters of Electric Permittivity and Magnetic Permeability of the medium under study. In any case, Newton did not explain, in terms of more fundamental concepts and constructions, the dependence of the index of refraction for air/water upon color.

Informed Seeing

22° Under certain conditions, streaks of green and purple appear at the lower edge of the peak of the rainbow. One refers to the streaks as Supernumerary Arcs. To the naive observer, these arcs are simply a part of the sweep of color in the rainbow. To the informed observer, however, they pose a new problem. The arcs have no "place" in the Cartesian/Newtonian theory. To explain the supernumerary arcs, one must invoke not the Particle Model but the Wave Model of Light, one must investigate the optical phenomenon of Interference, and one must analyze the Perception of Color in the Eye/Mind of the observer.

Existence/Uniqueness

23° One may rightly ask whether the rainbow "exists," and, if so, whether it is "unique." For a given observer, the observed rainbow is not an object but a conical assembly of directions. For distinct observers, the observed rainbows are distinct. Unlike the circumstance in which such sensory impressions as tree-like legs and a snake-like trunk could be explained to a group of blind men as aspects of the same underlying Elephant, for the aggregate of sensory impressions to which we refer as the rainbow, there is no underlying common object, unless one is content to declare it to be a State of the Atmosphere. The rainbow shares in the subtlety of distinctions between Matter and Light, between Thing and Process.

References

24° Very often, a secondary rainbow appears in the sky, above the primary bow. Can one adapt the cartesian explanation to the secondary bow? This and many other questions are treated in the following books:

The Rainbow: From Myth to Mathematics, Carl B. Boyer, 1987a
Geometry Civilized, J. L. Heilbron, 1998a
Light and Color in the Outdoors, M. G. J. Minnaert, 1993a
Introduction to Meteorological Optics, R. A. R. Tricker, 1970a

By study of these books, one will be able to form answers to such questions as the following:

(◦) Why does one see just two rainbows?
(◦) Does the size of the raindrops effect the appearance of the rainbow?
(◦) Should one expect to see a rainbow in a shower of sulphuric acid on Venus?
(◦) ...... in a shower of lead sulphate on Earth?
(◦) Would an Orca see a rainbow in a quiet sea, formed in a rising shower of air bubbles?