MATHEMATICS 391
ASSIGNMENT 10
Due: November 18, 2015

01° Let n be a positive integer. Let K be a finite set. Let $\mathcal{U}$ be an injective mapping carrying K to $\mathbf{R}^n$. Of course, $\mathcal{U}$ defines a list:

$$\mathcal{U}(k) \qquad (k \in K)$$

of distinct vectors in R^n. Let Π be the *polytope* generated by $\mathcal{U}$. By definition, the vectors in Π stand as follows:

$$(*) \qquad X = \sum_{k \in K} a(k)\mathcal{U}(k)$$

where a is any mapping carrying K to $[0, 1]$ for which:

$$\sum_{k \in K} a(k) = 1$$

Verify that Π is convex. We mean to say that, for any vectors Y and Z in Π and for any b and c in $[0, 1]$, if $b + c = 1$ then $V = aY + bZ$ is in Π.

02• For particular instances of $\mathcal{U}$, it may happen that there exist vectors X for which the coefficients in $(*)$ are not unique. Describe an example. For polytopes in general, there is no perfect remedy. However, there is a satisfactory remedy. One may show that there is a subset L of K such that the restriction $\mathcal{V}$ of $\mathcal{U}$ to L generates the same polytope Π, while, for any subset M of L, if the restriction $\mathcal{W}$ of $\mathcal{U}$ to M generates Π then $M = L$. Do so. One may say that $\mathcal{V}$ generates Π *minimally*. As we shall see, L is unique and $\mathcal{V}$ defines a very special list:

$$(\circ) \qquad \mathcal{V}(\ell) \qquad (\ell \in L)$$

of distinct vectors in Π.

03• Let V be a vector in Π. One says that V is a *vertex* of Π iff, for any vectors Y and Z in Π and for any b and c in $(0, 1)$, if $b + c = 1$ and if $V = bY + cZ$ then $Y = Z$. One might say that V is a vertex of Π iff it cannot be presented as a vector interior to a line segment in Π. Show that the vertices of Π are precisely the vectors in the list $(\circ)$. Now explain why the aforementioned subset L of K is unique.

$04^{\bullet}$ For certain instances of $\mathcal{U}$, the foregoing satisfactory remedy proves to be perfect. Following reduction to L and $\mathcal{V}$ (if necessary), the coefficients in $(*)$:

$$(*) \qquad Y = \sum_{\ell \in L} b(\ell)\mathcal{V}(\ell)$$

are unique. The latter assertion means that, for any vector Y in Π and for any mappings b' and b'' carrying L to $[0, 1]$, if:

$$\sum_{\ell \in L} b'(\ell) = 1, \quad \sum_{\ell \in L} b''(\ell) = 1, \quad \text{and} \quad \sum_{\ell \in L} b'(\ell) = Y = \sum_{\ell \in L} b''(\ell)$$

then $b' = b''$. In this context, one refers to the polytope Π as a *simplex*.

$05^{\bullet}$ Let $n = 4$ and let:

$$K = \{1, 2, 3, 4, 5, 6\}$$

Let $\mathcal{U}$ be defined as follows:

$$\mathcal{U}(1) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \mathcal{U}(2) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathcal{U}(3) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ \mathcal{U}(4) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and:

$$\mathcal{U}(5) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \mathcal{U}(6) = \frac{1}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

Describe the polytope defined by $\mathcal{U}$. Is it a simplex?