MATHEMATICS 391
ASSIGNMENT 7: Poincaré and Kac
Due: October 28, 2015

$01^{\bullet}$ Let:

$$(X, \mathcal{B}, \mu)$$

be a *normalized* measure space. By definition $\mathcal{B}$ is a σ-algebra of subsets of X, μ is a measure defined on $\mathcal{B}$, and $\mu(X) = 1$. Let F be a measurable mapping carrying X to itself. We mean to say that, for each set B in $\mathcal{B}$, $F^{-1}(B)$ is a set in $\mathcal{B}$. Let F preserve μ:

$$F_*(\mu) = \mu$$

so that, for each set B in $\mathcal{B}$:

$$\mu(F^{-1}(B)) = \mu(B)$$

One refers to:

$$(X, \mathcal{B}, \mu, F)$$

as a (discrete) *dynamical system*. For each x in X, one refers to:

$$x = F^0(x),\ x_1 = F^1(x),\ x_2 = F^2(x),\ \cdots,\ x_j = F^j(x),\ \cdots$$

as the *orbit* of x in X. Let A be any set in $\mathcal{B}$ for which $0 < \mu(A)$. Let:

$$B = A \cup F^{-1}(A) \cup F^{-2}(A) \cup \cdots \cup F^{-j}(A) \cup \cdots$$

Show that:

$$F^{-1}(B) = F^{-1}(A) \cup F^{-2}(A) \cup \cdots \cup F^{-j}(A) \cup \cdots$$

Obviously, $F^{-1}(B) \subseteq B$. Let:

$$C = B \backslash F^{-1}(B)$$

Show that C is composed of the points in A which, under the action of F, *leave* A and never *return*. Show that:

$$\mu(C) = 0$$

Let $A^* = A \backslash C$. Of course $A^* \subseteq A$. Show that:

$$\mu(A^*) = \mu(A)$$

Show that every point in A^* returns at least once to A. This assertion is the (weak) RECURRENCE THEOREM of Poincaré. Now let:

$$A^{**} = (A^* \backslash F^{-1}(C)) \cap (A^* \backslash F^{-2}(C)) \cap \cdots \cap (A^* \backslash F^{-j}(C)) \cap \cdots$$

Of course, $A^{**} \subseteq A^*$. Show that:

$$\mu(A^{**}) = \mu(A^*) = \mu(A)$$

Show that every point in A^{**} returns infinitely often to A^{**}. This assertion is the (strong) RECURRENCE THEOREM of Poincaré. For each point x in A^{**}, let $\tau(x)$ be the smallest among all positive integers j for which $F^j(x) \in A^{**}$. By definition:

$$F^{\tau(x)}(x) \in A^{**}$$

One refers to $\tau(x)$ as the *first return time* to A^{**} for x. Now let the (discrete) dynamical system $(X, \mathcal{B}, \mu, F)$ be ERGODIC, which is to say that:

$(\bullet)$ for any set E in $\mathcal{B}$, if $F^{-1}(E) = E$ then $\mu(E) = 0$ or $\mu(E) = 1$

For later reference, note that, as an immediate consequence of condition $(\bullet)$, if $0 < \mu(E)$ and if $F(E) \subseteq E$ then $\mu(E) = 1$. (To prove that it is so, one need only form the set:

$$\hat{E} = \bigcup_{j=0}^{\infty} F^{-j}(E) \qquad (E \subseteq F^{-1}(E) \subseteq F^{-2}(E) \subseteq \cdots)$$

then observe that $0 < \mu(E) = \mu(\hat{E})$ and that $F^{-1}(\hat{E}) = \hat{E}$, so that $\mu(E) = \mu(\hat{E}) = 1$.) At this point, prove that:

$$\frac{1}{\mu(A^{**})} \int_{A^{**}} \tau(x)\mu(dx) = \frac{1}{\mu(A^{**})}$$

This assertion is the AVERAGE RETURN TIME THEOREM of Kac. One may infer that the smaller the measure of A^{**} the longer the average return time to A^{**}. The proof is slippery. Simplify matters by presuming that F is invertible. Start with the subsets A_k of A^{**}:

$$A_k = \{x \in A^{**} : \tau(x) = k\}$$

where k is any positive integer. Now prove the Theorem of Kac by showing that:

$$(\kappa) \qquad \sum_{j=1}^{\infty} k\mu(A_k) = 1$$

To that end, form the sets:

$$E_{k\ell} = F^{\ell}(A_k) \qquad (k \in \mathbf{Z}^+,\ 0 \le \ell < k)$$

Show that the foregoing sets are mutually disjoint. (For this step, the presumption that F is invertible proves crucial.) Let E be the union of all of them:

$$E = \bigcup_{k=1}^{\infty} \bigcup_{\ell=0}^{k-1} E_{k\ell}$$

By examining the following display:

$$\begin{array}{cccccc}
A_1 & A_2 & A_3 & A_4 & A_5 & \cdots \\
 & F^1(A_2) & F^1(A_3) & F^1(A_4) & F^1(A_5) & \cdots \\
 & & F^2(A_3) & F^2(A_4) & F^2(A_5) & \cdots \\
 & & & F^3(A_4) & F^3(A_5) & \cdots \\
 & & & & F^4(A_5) & \cdots \\
 & & & & \vdots &
\end{array}$$

you should see that:

$$\mu(E) = \sum_{j=1}^{\infty} k\mu(A_k) = \int_{A^{**}} \tau(x)\mu(dx)$$

Obviously, $F(E) \subseteq E$. By the remark following condition (•) (which introduces the definition of ERGODIC for a dynamical system), you may infer that $\mu(E) = 1$. The prove of the Theorem of Kac is complete.