MATHEMATICS 391
ASSIGNMENT 5: The Weak Law of Large Numbers
Due: October 7, 2015

01° Let $(X, \mathcal{A}, \pi)$ be a probability space. Let n be a positive integer and let:

$$F_1, F_2, \ \ldots \ , F_n$$

be $\mathbf{R}$-valued random variables on X. For each index j $(1 \leq j \leq n)$, let m_j be the mean of F_j:

$$m_j = \int_X F_j(x)\mu(dx)$$

and let s_j^2 be the variance:

$$s_j^2 = \int_X (F_j(x) - m_j)^2 \mu(dx)$$

Let the random variables be *uncorrelated*, in the sense that, for any indices j and k $(1 \leq j < k \leq n)$:

$$\int_X (F_j(x) - m_j)(F_k(x) - m_k)\mu(dx) = 0$$

Let $\bar{F}$ be the mean of the random variables;

$$\bar{F} = \frac{1}{n}(F_1 + F_2 \ \cdots \ + F_n)$$

Let $\bar{m}$ be the mean of $\bar{F}$:

$$\bar{m} = \int_X \bar{F}(x)\mu(dx)$$

and let $\bar{s}^2$ be the variance:

$$\bar{s}^2 = \int_X (\bar{F}(x) - \bar{m})^2 \mu(dx)$$

Show that:

$$\bar{m} = \frac{1}{n}(m_1 + m_2 + \ \cdots \ + m_n)$$

and:

$$\bar{s}^2 = \frac{1}{n^2}(s_1^2 + s_2^2 + \ \cdots \ + s_n^2)$$

02° Let the random variables F_1, F_2, ... , and F_n have common mean, soit m. Note that:

$$\bar{m} = m$$

Le them have common variance, soit s^2. Note that:

$$\bar{s}^2 = \frac{1}{n} s^2$$

Let a be any positive number. Apply Chebychev's Inequality to show that:

$$\mu(\{x \in X : a \leq |\bar{F}(x) - m|\}) \leq \frac{s^2}{na^2}$$

Let ℓ be any positive integer and let $n = 100\ell^2$. Note that:

$$\mu(\{x \in X : \frac{s}{\ell} \leq |\bar{F}(x) - m|\}) \leq \frac{1}{100}$$

03° Let the random variables F_1, F_2, ... , and F_n be *independent*, which is to say that, for any borel sets:

$$E_1, E_2, \ \ldots \ , E_n$$

of $\mathbf{R}$:

$$\begin{aligned} &\mu(F_1^{-1}(E_1) \cap F_2^{-1}(E_2) \cap \ \cdots \ \cap F_n^{-1}(E_n)) \\ &\qquad\qquad = \mu(F_1^{-1}(E_1)\mu(F_2^{-1}(E_2) \ \cdots \ \mu(F_n^{-1}(E_n) \end{aligned}$$

In the lectures, we will prove (prove again, actually) that, for any indices j and k $(1 \leq j < k \leq n)$:

$$\int_X F_j(x)F_k(x)\mu(dx) = \int_X F_j(x)\mu(dx) \int_X F_k(x)\mu(dx)$$

Using the foregoing relations, prove that the random variables F_1, F_2, ... , and F_n are uncorrelated.