MATHEMATICS 391
ASSIGNMENT 3
Due: September 23, 2015

01• Let X be any set, let $\mathcal{B}$ be a borel algebra of subsets of X, and let μ be a probability measure on $\mathcal{B}$. Let:

$$A_1, A_2, \ \ldots \ , A_n, \ \ldots$$

be arbitrary sets in $\mathcal{B}$. One can easily verify that:

$$\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) - \mu(A_1 \cap A_2)$$

and that:

$$\begin{aligned} \mu(A_1 \cup & A_2 \cup A_3) \\ &= \mu(A_1) + \mu(A_2) + \mu(A_3) \\ &\quad - \mu(A_1 \cap A_2) - \mu(A_1 \cap A_3) - \mu(A_2 \cap A_3) \\ &\qquad + \mu(A_1 \cap A_2 \cap A_3) \end{aligned}$$

Prove (by induction) the Inclusion/Exclusion principle:

$$\mu(A_1 \cup A_2 \cup \ \cdots \ \cup A_n) = \sum_{\sigma} (-1)^{|\sigma|+1} \mu(A_{\sigma_1} \cap A_{\sigma_2} \cap \ \cdots \ \cap A_{\sigma_k})$$

where σ is any sequence of integers of the form:

$$1 \le \sigma_1 < \sigma_2 < \ \cdots \ < \sigma_k \le n$$

and where:

$$|\sigma| := k$$

Now let q be any positive integer and let X be the (finite) set of all permutations π:

$$\pi = (\pi_1, \pi_2, \ \cdots \ , \pi_q)$$

of the positive integers:

$$1, 2, 3, \ \ldots \ , q$$

Let μ be the uniform probability measure on $\mathcal{P}(X)$. For each positive integer p $(1 \le p \le q)$, let A_p be the subset of X consisting of all permutations π for which $\pi_p = p$. Applying the Inclusion/Exclusion principle, show that:

$$\lim_{q \to \infty} \mu(A_1 \cup A_2 \cup \ \cdots \ \cup A_q) = 1 - \frac{1}{e} = 0.63245...$$

$02^{\bullet}$ The King stands on the southwestmost square of his chessboard. He moves one square north with probability 0.25, one square east with probability 0.25, or one square (diagonally) northeast with probability 0.50. He continues to move until he reaches either the northmost rank or the eastmost file, at which point he stops. What is the probability that he reaches the northeastmost square? To develop the solution, you might start by considering smaller boards than the conventional 8 by 8 chessboard.

$03^{\bullet}$ Let X be the (finite) set composed of the ordered pairs of integers:

$$(j,k) \qquad (1 \leq j \leq 6,\ 1 \leq k \leq 6)$$

Let μ be the probability measure on $\mathcal{P}(X)$ defined by the condition:

$$\mu(\{(\ell,\ell)\}) := \frac{1}{6} \qquad (1 \leq \ell \leq 6)$$

Let Y be the (finite) set comprised of the integers:

$$1,2,3,4,5,6$$

Let π' and π'' be the mappings carrying X to Y defined as follows:

$$\pi'((j,k)) := j, \quad \pi''((j,k)) := k \qquad ((j,k) \in X)$$

Of course, they are random variables. Describe the distributions for π' and π''. That is, describe the probability measures $\nu' := (\pi')_*(\mu)$ and $\nu'' := (\pi'')_*(\mu)$ on $\mathcal{P}(Y)$. Are these random variables independent?

$04^{\bullet}$ Let k and n be any positive integers and let:

$$x_1, x_2, \ \ldots \ , x_k$$

be any real numbers. By generalization of the common binomial expansion, we obtain the following *multinomial expansion*:

$$(1) \qquad (x_1 + x_2 + \ \cdots \ + x_k)^n = \sum_{\bar{n} \in Y} \frac{n!}{n_1! n_2! \ \cdots \ n_k!} x_1^{n_1} x_2^{n_2} \ \cdots \ x_k^{n_k}$$

where Y is the set of all k-tuples $\bar{n}$:

$$\bar{n} = (n_1, n_2, \ \ldots \ , n_k)$$

of nonnegative integers for which:

$$n_1 + n_2 + \ \cdots \ + n_k = n$$

Two basic facts spring from this relation. First, if:

$$x_1 = 1,\ x_2 = 1,\ \dots\ ,\ x_k = 1$$

then:

$$k^n = \sum_{\bar{n} \in Y} \frac{n!}{n_1! n_2!\ \cdots\ n_k!} \tag{2}$$

Second, if:

$$x_1 + x_2 +\ \cdots\ + x_k = 1$$

then:

$$1 = \sum_{\bar{n} \in Y} \frac{n!}{n_1! n_2!\ \cdots\ n_k!}\, x_1^{n_1} x_2^{n_2}\ \cdots\ x_k^{n_k} \tag{3}$$

In particular, if:

$$x_1 = \frac{1}{k},\ x_2 = \frac{1}{k},\ \dots\ ,\ x_k = \frac{1}{k}$$

then:

$$1 = \sum_{\bar{n} \in Y} \frac{n!}{n_1! n_2!\ \cdots\ n_k!}\, k^{-n} \tag{4}$$

consistent with relation (2). Now let us imagine a "die" having k "faces." Let:

$$x_1, x_2,\ \dots\ , x_k$$

be nonnegative real numbers for which:

$$x_1 + x_2 +\ \cdots\ + x_k = 1$$

Let j be any index ($1 \leq j \leq k$). Let us imagine that, when rolled, the die presents its j-th face with probability x_j. Let us make n distinguishable copies of the die and let us roll them out on the table. The various possible outcomes of this "experiment" can be represented as follows:

$$\bar{a} = (a_1, a_2,\ \dots\ , a_n)$$

where, for each index m ($1 \leq m \leq n$), a_m is an integer between 1 and k:

$$1 \leq a_m \leq k$$

representing the upturned face on the m-th die. Of course, there are k^n such outcomes. Let X be the set of all of them. For each $\bar{a}$ in X, we may form the corresponding $\bar{n}$ in Y such that, for each index j ($1 \leq j \leq k$), n_j is the number of indices m $(1 \leq m \leq n)$ for which $a_m = j$. By this observation, we may introduce the mapping F carrying X to Y defined as follows:

$$F(\bar{a}) := \bar{n} \qquad (\bar{a} \in X)$$

As an illustration, let us take k to be 6, n to be 10, and $\bar{a}$ to be:

$$\bar{a} = (1, 6, 2, 6, 5, 5, 1, 5, 3, 3)$$

We obtain:

$$\bar{n} = F(\bar{a}) = (2, 1, 2, 0, 3, 2)$$

Let us imagine that we design our experiment so that the various faces of the dice come up "independently." We may describe this situation formally by introducing the following probability measure μ on $\mathcal{P}(X)$:

$$\mu(\{\bar{a}\}) = x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} \qquad (\bar{a} \in X) \tag{5}$$

where $\bar{n} := F(\bar{a})$. We may then (as usual) form the probability measure $\nu := F_*(\mu)$ on $\mathcal{P}(Y)$. Show that:

$$\nu(\{\bar{n}\}) = \frac{n!}{n_1! n_2! \cdots n_k!} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} \qquad (\bar{n} \in Y) \tag{6}$$

One refers to ν as the *multinomial* probability measure on $\mathcal{P}(Y)$, defined by the probabilities:

$$x_1, x_2, \ldots, x_k$$

It is the distribution of the foregoing random variable F. Let us consider a special case. Let $k = 6$, $n = 10$, and:

$$x_1 = \frac{1}{6},\ x_2 = \frac{1}{6},\ x_3 = \frac{1}{6},\ x_4 = \frac{1}{6},\ x_5 = \frac{1}{6},\ x_6 = \frac{1}{6}$$

In this case, Y consists of the 6-tuples:

$$(n_1, n_2, n_3, n_4, n_5, n_6)$$

of nonnegative integers for which:

$$n_1 + n_2 + + n_3 + n_4 + n_5 + n_6 = 10$$

We find that:

$$\nu(\{(n_1, n_2, n_3, n_4, n_5, n_6)\}) = \frac{10!}{n_1! n_2! n_3! n_4! n_5! n_6!} 6^{-10} \tag{7}$$

where:

$$(n_1, n_2, n_3, n_4, n_5, n_6)$$

is any member of Y. Let B be the subset of Y consisting of all:

$$(n_1, n_2, n_3, n_4, n_5, n_6)$$

for which $n_6 = 5$. To compute $\nu(B)$, we first introduce the set Z comprised of all 5-tuples:

$$(n_1, n_2, n_3, n_4, n_5)$$

of nonnegative integers for which:

$$n_1 + n_2 + n_3 + n_4 + n_5 = 5$$

Then:

$$\begin{aligned}
\nu(B) :=& \sum_{(n_1,n_2,n_3,n_4,n_5,n_6)\in B} \frac{10!}{n_1!n_2!n_3!n_4!n_5!n_6!} 6^{-10} \\
&= 10!5!^{-2}6^{-10} \sum_{(n_1,n_2,n_3,n_4,n_5)\in Z} \frac{5!}{n_1!n_2!n_3!n_4!n_5!} \\
&= 10!5!^{-2}6^{-10}5^5
\end{aligned}$$

by relation (2). Hence:

$$\nu(B) = 0.0130238$$

Let us continue to consider the foregoing special case. Let C be the subset of Y consisting of all:

$$(n_1, n_2, n_3, n_4, n_5, n_6)$$

for which:

$$n_1 = 0 \text{ or } n_2 = 0 \text{ or } n_3 = 0 \text{ or } n_4 = 0 \text{ or } n_5 = 0 \text{ or } n_6 = 0$$

By applying the Inclusion/Exclusion Principle, show that:

$$\begin{aligned}
\nu(C) &= 6^{-10}(\binom{6}{1}5^{10} - \binom{6}{2}4^{10} + \binom{6}{3}3^{10} - \binom{6}{4}2^{10} + \binom{6}{5}1^{10}) \\
&= 0.728188
\end{aligned}$$