MATHEMATICS 331
EXAMINATION
Due: Wednesday, May 13, 2015, NOON, Library 306

01° Describe an example of a finite dimensional linear space $\mathbf{V}$ and two linear mappings L_1 and L_2 in $\mathbf{L}(\mathbf{V})$ such that $L_1 L_2 = 0$ while $L_2 L_1 \neq 0$.

02° Let n be a positive integer. Let $\mathbf{V}$ be a finite dimensional linear space for which the dimension is n. Let Λ be a linear mapping in $\mathbf{V}^*$, that is, a linear functional defined on $\mathbf{V}$. Granted that $\Lambda \neq 0$, find the dimension of the kernel of Λ.

03° Let $\mathbf{V}$ be a finite dimensional linear space and let Π be a projection mapping in $\mathbf{L}(\mathbf{V})$. . We mean to say that Π is linear and that $\Pi^2 = \Pi\,\Pi = \Pi$. Show that the list:

$$\mathcal{U}: \quad ker(\Pi),\, ran(\Pi)$$

of linear subspaces of $\mathbf{V}$ defines a direct sum decomposition of $\mathbf{V}$:

$$\mathbf{V} = ker(L) \oplus ran(L)$$

04° Let a, b, and c be any numbers. Show that:

$$det \begin{pmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{pmatrix} = abc(b-a)(c-a)(c-b)$$

05° Let L be the matrix of real numbers having 5 rows and 3 columns and let Y be the vector having 5 rows and 1 column, defined as follows:

$$L = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}, \qquad Y = \begin{pmatrix} -1 \\ 1 \\ 6 \\ 1 \\ -1 \end{pmatrix}$$

Find the vector X having 3 rows and 1 column:

$$X = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

for which:

$$\|LX - Y\|^2$$

is least among all such vectors.

06° Let **V** be a finite dimensional linear space and let L be a linear mapping in $\mathbf{L}(\mathbf{V})$. Show that if L is both diagonalizable and nilpotent then $L = 0$.

07° Let **V** be a pdo geometry and let L be a linear mapping in $\mathbf{L}(\mathbf{V})$. Show that there exist a nonnegative self adjoint linear mapping S and an orthogonal linear mapping U in $\mathbf{L}(\mathbf{V})$ such that $L = SU$.

08° Let **V** be the linear space consisting of all matrices of real numbers having 3 rows and 3 columns:

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

Determine the dimension of **V**. Let T be the linear functional in $\mathbf{V}^*$ defined as follows:

$$T(M) = m_{11} + m_{22} + m_{33}$$

where M is any matrix in **V**. Let Γ be the bilinear form defined on **V** as follows:

$$\Gamma(M_1, M_2) \equiv \langle\!\langle\, M_1, M_2 \,\rangle\!\rangle = T(M_2^* M_1)$$

where M_1 and M_2 are any matrices in **V**. Show that Γ is orthogonal and positive definite.

09• Snow White distributed 21 liters of whole milk among the seven dwarfs. They were carrying pails. The first dwarf (Doc) then distributed the contents of his pail evenly to the pails of the other six dwarfs. Then the second dwarf (Happy) did the same, and so on in turn: Sleepy, Sneezy, Bashful, and Dopey. After the seventh dwarf (Grumpy) distributed the contents of his pail evenly to the other six dwarfs, Snow White found that each dwarf had exactly the same amount of milk in his pail as she had given him at the start. What was the initial distribution of the milk? Of course, we should all note that Grumpy had no milk at the end, hence no milk at the beginning.