MATHEMATICS 331
ASSIGNMENT 10
Due: April 16, 2015

01° Let S be the matrix defined by the following quadratic array:

$$S = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Of course, S is a linear mapping carrying $\mathbf{R}^3$ to itself. Show that S is self adjoint. Find the characteristic values for S:

$$a_1, a_2, a_3$$

Find an orthonormal basis for $\mathbf{R}^3$:

$$C_1, C_2, C_3$$

which serves to diagonalize S:

$$S(C_1) = a_1 C_1,\ S(C_2) = a_2 C_2,\ S(C_3) = a_3 C_3$$

02° In context of the foregoing problem, let U be the orthogonal linear mapping carrying $\mathbf{R}^3$ to itself, defined by the following relations:

$$U(E_1) = C_1,\ U(E_2) = C_2,\ U(E_3) = C_3$$

We refer to U as the "change of basis" matrix: Verify that:

$$U^* U = I = U U^* \quad \text{hence} \quad U^* = U^{-1}$$

In turn, verify that:

$$U^* S U = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$$

We say that the change of basis defined by U serves to "diagonalize" S. We infer that the "geometric properties" of S and $S^* = U^* S U$ are the same. To understand the sense of this inference, one should verify that, for any X and Y in $\mathbf{R}^3$:

$$\langle\!\langle\, U(X), U(Y) \,\rangle\!\rangle = \langle\!\langle\, X, Y \,\rangle\!\rangle$$

03° Show that, for any orthogonal linear mapping:

$$det(U) = \pm 1$$

We say that U is a *rotation* iff $det(U) = 1$. We say that U is a *reflection* iff $det(U) = -1$.

04° Let N be a member of $\mathbf{R}^3$ for which:

$$\langle\!\langle N, N \rangle\!\rangle = 1$$

Let U be the linear mapping carrying $\mathbf{R}^3$ to itself, defined as follows:

$$U(X) = X - 2\langle\!\langle X, N \rangle\!\rangle N$$

where X is any member of $\mathbf{R}^3$. Show that U is a reflection. To that end, study the cases in which $\langle\!\langle X, N \rangle\!\rangle = 0$ and $X = \xi N$.

05• Again let N be a member of $\mathbf{R}^3$ for which:

$$\langle\!\langle N, N \rangle\!\rangle = 1$$

Let A be the corresponding matrix:

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}, \quad \text{where} \quad N = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Of course, $a^2 + b^2 + c^2 = 1$. Show that, for any real number t:

$$V(t) = exp(tA) = I + sin(t)A + (1 - cos(t))A^2$$

Show that $V(t)$ is a rotation. The *axis* of $V(t)$ is N:

$$V(t)N = N$$

because $AN = 0$.