MATHEMATICS 322
ASSIGNMENT 10
Due: November 18, 2015

01$^{\bullet}$ We define the Associated Legendre Functions $P_{l,m}$ on the open interval $(-1,1)$ by the following relation:

$$P_{l.m}(u) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi}(u + i\sqrt{1-u^2}\,cos\,t)^{\ell}e^{-imt}dt$$

In this context, ℓ and m are integers for which $0 \leq \ell$ and $-\ell \leq m \leq \ell$. With reference to the Theory of Fourier Series, show that:

$$(u + i\sqrt{1-u^2}\,cos\,t)^{\ell} = \sum_{m=-\ell}^{\ell} P_{\ell,m}(u)e^{imt}$$

Why is the sum finite? Verify that:

$$P_{\ell,-m}(u) = P_{\ell,m}(u)$$

Let us denote $P_{\ell,0}$ by the simpler symbol P_{ℓ}:

$$P_{\ell}(u) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi}(u + i\sqrt{1-u^2}\,cos\,t)^{\ell}dt$$

Show that (the Legendre Function) P_{ℓ} satisfies the Legendre Equation:

$$(L) \qquad (1-u^2)W^{\circ\circ}(u) - 2uW^{\circ}(u) + \ell(\ell+1)W(u) = 0$$

To do so, you might want to introduce the changes of variables:

$$Q_{\ell}(\theta) = P_{\ell}(sin\,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(sin\,\theta + i\,cos\,\theta\,cos\,t)^{\ell}dt$$

and $V(\theta) = W(sin\,\theta)$, so that relation (L) becomes the following relation:

$$(\bar{L}) \qquad V^{\circ\circ}(\theta) - tan\,\theta\,V^{\circ}(\theta) + \ell(\ell+1)V(\theta) = 0$$

02$^{\bullet}$ Show that:

$$(D) \qquad P_{\ell,m}(u) = (1-u^2)^{m/2}\frac{d^m}{du^m}P_{\ell}(u) \qquad (0 \leq m \leq \ell)$$

03• Show that $P_{\ell,m}$ satisfies the Associated Legendre Equation:

$$(A) \qquad (1-u^2)W^{\circ\circ}(u) - 2uW^{\circ}(u) + \big(\ell(\ell+1) - \frac{m^2}{1-u^2}\big)W(u) = 0$$

Patient application of (L) and (D) will yield the result.

04• Memorize the following display of the Spherical Harmonics:

$$(Y) \qquad Y_{\ell,m}(\phi,\theta) \equiv P_{\ell,m}(sin\,\theta)e^{im\phi}$$

Of course, $0 \le \ell$ and $-\ell \le m \le \ell$. The Spherical Harmonics are eigenfunctions for the Spherical Laplacian:

$$-(\frac{1}{cos^2\theta}\frac{\partial^2}{\partial\phi^2} - tan\,\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2})Y_{\ell,m}(\phi,\theta) = \ell(\ell+1)Y_{\ell,m}(\phi,\theta)$$

Suitably normalized:

$$(\bar{Y}) \qquad \mathcal{Y}_{\ell,m}(\phi,\theta) \equiv \sqrt{(2\ell+1)\frac{(\ell-|m|)!}{(\ell+|m|)!}}P_{\ell,m}(sin\,\theta)e^{im\phi}$$

they form an orthonormal basis for the inner product space $\mathbf{L}^2(\mathbf{S}^2)$, composed of all complex valued square integrable (borel) functions defined on $\mathbf{S}^2$ and supplied with the inner product and the corresponding norm:

$$\langle\!\langle f, g \rangle\!\rangle \equiv \frac{1}{4\pi}\int\!\!\int_{\mathbf{S}^2} f(\phi,\theta)\overline{g(\phi,\theta)}cos\,\theta\,d\phi\,d\theta$$

$$\langle\!\langle h \rangle\!\rangle \equiv \sqrt{\langle\!\langle h, h \rangle\!\rangle}$$

To be clear, let us emphasize that ϕ and θ stand for longitude and latitude, respectively.