**MATHEMATICS 321**
ASSIGNMENT 6: SOLUTIONS
Due: October 14, 2015

01° Let **D** be the unit disk in **C**, consisting of all complex numbers $\zeta$ such that $|\zeta| \leq 1$. Let $\mathbf{A}'$ be the subalgebra of $\mathbf{C}(\mathbf{D})$ consisting of all functions of the form:

$$f(\zeta) = \sum_{j=0}^{n} \alpha_j \zeta^j \qquad (\zeta \in \mathbf{D})$$

where $n$ is any nonnegative integer and where the various $\alpha_j$ are any complex numbers. Let $\mathbf{A}''$ be the subalgebra of $\mathbf{C}(\mathbf{D})$ consisting of all functions of the form:

$$g(\zeta) = \sum_{j=0}^{n} \sum_{k=0}^{n} \beta_{jk} \zeta^j \zeta^{*k} \qquad (\zeta \in \mathbf{D})$$

where $n$ is any nonnegative integer and where the various $\beta_{jk}$ are any complex numbers. Verify that both $\mathbf{A}'$ and $\mathbf{A}''$ separate points in **D**. Show that $\mathbf{A}''$ is involutory while $\mathbf{A}'$ is not. Prove that $\mathbf{A}''$ is dense in $\mathbf{C}(\mathbf{D})$ while $\mathbf{A}'$ is not.

[Obviously, $\mathbf{A}'$ meets the conditions of the hypothesis of Stone's Theorem, with the possible exception of the condition that it be involutory. We infer that if, in fact, $\mathbf{A}'$ is involutory then (by Stone's Theorem) $\mathbf{C}(\mathbf{D}) = clo(\mathbf{A}')$. Let $h$ be the function in $\mathbf{C}(\mathbf{D})$ defined as follows:

$$h(\zeta) = \zeta^*$$

where $\zeta$ is any member of **D**. We contend that $h$ is not contained in $clo(\mathbf{A}')$. Of course, it would follow that $clo(\mathbf{A}') \neq \mathbf{C}(\mathbf{D})$, hence that $\mathbf{A}'$ is not involutory. Let us suppose to the contrary that $h$ is in $clo(\mathbf{A}')$. Under this supposition, we may introduce a function:

$$f(\zeta) = \sum_{j=0}^{n} \alpha_j \zeta^j \qquad (\zeta \in \mathbf{D})$$

in $\mathbf{A}'$ such that:

$$|\zeta^* - f(\zeta)| \leq \frac{1}{2} \qquad (\zeta \in \mathbf{D})$$

Hence:

$$|1 - \sum_{j=0}^{n} \alpha_j \zeta^{j+1}| \leq \frac{1}{2} \qquad (|\zeta| = 1)$$

That is:

$$|1 - \sum_{j=0}^{n} \alpha_j exp(i(j+1)\theta)| \leq \frac{1}{2} \qquad (0 \leq \theta < 2\pi)$$

At this point, let us recall that:

$$\frac{1}{2\pi}\int_0^{2\pi} exp(ik\theta)d\theta = \begin{cases} 0 & \text{if } k \neq 0 \\ 1 & \text{if } k = 0 \end{cases}$$

Now, from the preceding inequality, we would obtain:

$$1 \leq \frac{1}{2}$$

a contradiction. Consequently, our supposition is false. Therefore, our contention is true. ]

02° Let $X_1$ and $X_2$ be compact spaces. Let $f$ be a complex valued function defined and continuous on $X_1 \times X_2$. Let $r$ be any positive number. Show that there are a nonnegative integer $\ell$, complex valued functions:

$$g_j \qquad (0 \leq j \leq \ell)$$

defined and continuous on $X_1$, and complex valued functions:

$$h_j \qquad (0 \leq j \leq \ell)$$

defined and continuous on $X_2$ such that, for each $(\xi, \eta)$ in $X_1 \times X_2$:

$$\big|f(\xi,\eta) - \sum_{j=0}^{\ell} g_j(\xi)h_j(\eta)\big| \leq r$$

[ The various functions of the form:

$$\sum_{j=0}^{\ell} g_j(\xi)h_j(\eta)$$

compose an involutory subalgebra $\mathbf{A}$ of $\mathbf{C}(X_1 \times X_2)$. We contend that $\mathbf{A}$ separates points in $X_1 \times X_2$. To prove our contention, we appeal to the following more general result. Let $X$ be a metric space, with metric $d$, and let $w$ be a point in $X$. Let $f_w$ be the (real valued) function defined on $X$ as follows:

$$f_w(z) = d(z, w)$$

where $z$ is any point in $X$. By the triangle inequality:

$$d(x,w) \leq d(x,y) + d(y,w), \quad d(y,w) \leq d(y,x) + d(x,w)$$

where $x$ and $y$ are any points in $X$. Hence:

$$|f_w(x) - f_w(y)| \leq d(x, y)$$

It follows that $f$ is continuous. Finally, let $u$ and $v$ be any points in $X$ for which $u \neq v$. Obviously, $f_v$ separates $u$ and $v$:

$$0 < d(u, v) = f_v(u), \quad f_v(v) = 0$$

Now our contention follows easily. ]

03° Let $\mathbf{T}$ be the unit circle in $\mathbf{C}$ consisting of all complex numbers $\tau$ such that $|\tau| = 1$. Let $\mathbf{A}$ be the subalgebra of $\mathbf{C}(\mathbf{T})$ consisting of all functions of the form:

$$f(\tau) = \sum_{j=-n}^{n} \alpha_j \tau^j = \sum_{j=-n}^{n} \alpha_j e^{ij\theta} \qquad (\tau = e^{i\theta})$$

where $n$ is any nonnegative integer and where the various $\alpha_j$ are any complex numbers. Verify that $\mathbf{A}$ separates points in $\mathbf{T}$. Show that $\mathbf{A}$ is involutory. Conclude that $\mathbf{A}$ is dense in $\mathbf{C}(\mathbf{T})$.

04• With reference to the foregoing example, we describe an interesting fact. Let $f$ be any function in $\mathbf{C}(\mathbf{T})$. We form the Fourier Coefficients of $f$ as follows:

$$\gamma_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-ij\theta} d\theta$$

We form the corresponding Fourier Sequences:

$$s_n(e^{i\theta}) = \sum_{j=-n}^{n} \gamma_j e^{ij\theta}, \quad \sigma_m(e^{i\theta}) = \frac{1}{m} \sum_{n=0}^{m-1} s_n(e^{i\theta})$$

In general, $s$ represents $f$ rather loosely. It converges to $f$ in the Integral Metric, a rather weak condition. However, $\sigma$ converges to $f$ uniformly.