MATHEMATICS 321
ASSIGNMENT 2: Solutions
Due: September 16, 2015

01$^\bullet$ Let σ be the sequence in the metric space $\bar{\mathbf{M}}((0,1))$ defined as follows:

$$\sigma(j)(t) = t^j \qquad (j \in \mathbf{Z}^+, \ \ 0 < t < 1)$$

Let $\bar{0}$ be the zero function in $\bar{\mathbf{M}}((0,1))$:

$$\bar{0}(t) = 0 \qquad (0 < t < 1)$$

Show that, contrary to appearance, σ does not converge to $\bar{0}$. [In fact, σ does not converge.]

[Let δ be the (uniform) metric on $\bar{\mathbf{M}}((0,1))$. One need only note that:

$$\delta(\sigma(j), \bar{0}) = 1 \qquad (j \in \mathbf{Z}^+)$$

]

02$^\bullet$ Let $\mathbf{P}((0,1))$ be the subset of $\bar{\mathbf{M}}((0,1))$ consisting of all (complex valued) polynomial functions (restricted, of course, to $(0,1)$). Let D be the mapping carrying $\mathbf{P}((0,1))$ to itself, defined by differentiation:

$$D(p) = p' \qquad (p \in \mathbf{P}((0,1)))$$

Show that D is not continuous at $\bar{0}$. [In fact, D is not continuous at any polynomial in $\mathbf{P}((0,1))$.]

[Let σ be the sequence in $\mathbf{P}((0,1))$ defined as follows:

$$\sigma(j)(t) = \frac{1}{j} t^j \qquad (j \in \mathbf{Z}^+, \ \ 0 < t < 1)$$

Clearly:

$$\delta(\sigma(j), \bar{0}) \longrightarrow 0 \quad \text{hence} \quad \sigma \longrightarrow \bar{0}$$

but:

$$\delta(D(\sigma(j)), D(\bar{0})) \not\longrightarrow 0 \quad \text{hence} \quad D(\sigma) \longrightarrow D(\bar{0})$$

It follows that D is not continuous at $\bar{0}$.]

03$^\bullet$ Let $\mathbf{R}^2$ be supplied as usual with the cartesian metric. Describe a subset Y of $\mathbf{R}^2$ such that Y is open but:

$$Y \neq int(clo(Y))$$

$04^{\bullet}$ Let σ be a sequence in $\mathbf{R}$. Show that there must be a subsequence τ of σ such that τ is *monotone*. We mean to say that τ is increasing:

$$j \in \mathbf{Z}^{+} \Longrightarrow \tau(j) \leq \tau(j+1)$$

or that τ is decreasing:

$$j \in \mathbf{Z}^{+} \Longrightarrow \tau(j+1) \leq \tau(j)$$

[Let A be the subset of $\mathbf{Z}^{+}$ consisting of all positive integers j such that, for each positive integer k, if $j \leq k$ then $\sigma(k) \leq \sigma(j)$. The members of A are the *leaders*. It may happen that A is infinite:

$$A: \quad j_1 < j_2 < j_3 < \cdots$$

Obviously, the restriction of σ to A would be a decreasing subsequence of σ. It may happen that A is finite. In such a case, we may introduce a positive integer k such that, for each positive integer j, if $j \in A$ then $j < k$. Now, by definition, there must be a subset B of $\mathbf{Z}^{+}$:

$$B: \quad k = k_1 < k_2 < k_3 < \cdots$$

such that:

$$\sigma(k_1) < \sigma(k_2) < \sigma(k_3) < \cdots$$

Obviously, the restriction of σ to B would be an increasing subsequence of σ.]