EXAMINATION
MATHEMATICS 321
Due: L306, High Noon, Wednesday, December 16, 2015
NO LIVING SOURCES

01$^\bullet$ Let X be a metric space, with metric d. Let σ and τ be cauchy sequences in X. Let ρ be the sequence in $\mathbf{R}$, defined as follows:

$$\rho(j) = d(\sigma(j), \tau(j)) \qquad (j \in \mathbf{Z}^+)$$

Show that ρ is convergent. For a running start, note the following relations:

$$d(\sigma(k), \tau(k) \leq d(\sigma(k), \sigma(\ell)) + d(\sigma(\ell), \tau(\ell)) + d(\tau(\ell), \tau(k))$$
$$d(\sigma(k), \tau(k)) - d(\sigma(\ell), \tau(\ell)) \leq d(\sigma(k), \sigma(\ell)) + d(\tau(k), \tau(\ell))$$

02$^\bullet$ Let X be a metric space, with metric d. Let $\mathcal{K}$ be a sequence of nonempty compact subsets of X:

$$\mathcal{K}: \quad K_1, K_2, \ \ldots\ , K_j, \ \ldots$$

Let $\mathcal{K}$ be decreasing and let L be the intersection of the terms of $\mathcal{K}$:

$$\cdots\ K_j \subseteq \cdots \subseteq K_2 \subseteq K_1, \qquad L = \bigcap_{j=0}^{\infty} K_j$$

Show that $L \neq \emptyset$ and that $\mathcal{K}$ converges to L relative to the hausdorff metric δ:

$$\delta(K_j, L) \longrightarrow 0$$

03$^\bullet$ Let $\mathbf{B}$ be the closed ball in $\mathbf{R}^3$ centered at 0 with radius 1. Let F be a mapping carrying $\mathbf{B}$ to $\mathbf{R}^3$ which meets the conditions that, for any members x, y, and z in $\mathbf{B}$:

$$\|F(x)\| \leq 1 \quad \text{and} \quad \|F(y) - F(z)\| \leq \|y - z\|$$

Let σ be a number for which $0 < \sigma < 1$. Let $\mathbf{X}$ be the family:

$$\mathbf{X} = \mathbf{M}\big((-\sigma, \sigma), \mathbf{B}\big)$$

composed of all continuous mappings α carrying $(-\sigma, \sigma)$ to $\mathbf{B}$. We supply $\mathbf{X}$ with the uniform metric $\mathbf{d}$, as follows:

$$\mathbf{d}(\alpha_1, \alpha_2) = sup\{\|\alpha_1(t) - \alpha_2(t)\| \ : \ -\sigma < t < \sigma\}$$

where α_1 and α_2 are any mappings in $\mathbf{X}$. We take it for granted that $\mathbf{X}$ is a complete metric space. For each α in $\mathbf{X}$, let β be the mapping carrying $(-\sigma, \sigma)$ to $\mathbf{R}^3$, defined as follows:

$$\beta(t) = \int_0^t F(\alpha(u))du \qquad (-\sigma < t < \sigma)$$

Verify that β is in $\mathbf{X}$. Having done so, introduce the mapping $\mathbf{F}$ carrying $\mathbf{X}$ to itself, defined as follows:

$$\mathbf{F}(\alpha) = \beta \qquad (\alpha \in \mathbf{X})$$

Verify that $\mathbf{F}$ is a contraction mapping. In fact, verify that, for any members α_1 and α_2 of $\mathbf{X}$:

$$\mathbf{d}(\mathbf{F}(\alpha_1), \mathbf{F}(\alpha_2)) \leq \sigma \mathbf{d}(\alpha_1, \alpha_2)$$

By the Contraction Mapping Theorem, there is precisely one γ in $\mathbf{X}$ such that $\mathbf{F}(\gamma) = \gamma$. Obviously:

$$\gamma(0) = 0, \text{ and } \gamma^\circ(t) = F(\gamma(t)) \qquad (-\sigma < t < \sigma)$$

04$^\bullet$ Let X be a set, let $\mathcal{A}$ be a borel algebra of subsets of X, and let μ be a measure defined on $\mathcal{A}$ for which $\mu(X) = 1$. Let K be a compact convex subset of $\mathbf{C}$. Let f be a complex valued borel function defined on X such that the range of f is included in K. Show that f is integrable with respect to μ and that:

$$\int_X f(x)\mu(dx) \in K$$

05$^\bullet$ Let n be a positive integer. Let X be the set consisting of the integers x for which $0 \leq x \leq n$ and let Y be the unit interval:

$$X = \{0, 1, 2, \ldots, n\}, \qquad Y = [0, 1]$$

Let μ and ν be the uniform probability measures on X and Y, respectively. By definition, ν is the restriction of lebesgue measure λ to the unit interval, while μ stands as follows:

$$\mu(\{x\}) = \frac{1}{n+1} \qquad (x \in X)$$

Verify that:

$$\int_Y (n+1)\binom{n}{x} y^x (1-y)^{n-x} \nu(dy) = 1 \qquad (x \in X)$$

or take it for granted. Let Γ and Δ be the mappings carrying X to $M(Y)$ and Y to $M(X)$, respectively, characterized as follows:

$$\begin{aligned}\Gamma(x)(B) &= \int_B (n+1)\binom{n}{x} y^x(1-y)^{n-x}\nu(dy)\\ \Delta(y)(A) &= \sum_{x\in A}\binom{n}{x} y^x(1-y)^{n-x}\end{aligned}$$

where x and y are any members of X and Y, respectively, and where A and B are any borel subsets of X and Y, respectively. Verify that:

$$\Gamma_*(\mu) = \nu, \quad \Delta_*(\nu) = \mu$$

06• Let X be the set consisting of the eight members:

$$(j,k,\ell) \qquad (j,k,\ell \in \{0,1\})$$

The members of X are the vertices of the unit cube in $\mathbf{R}^3$. Let $\mathcal{A}$ be the borel algebra consisting of all subsets of X. Let μ be the measure on $\mathcal{A}$ defined by the following relations:

$$\mu(\{(0,0,0)\}) = \mu(\{(1,1,0)\}) = \mu(\{(1,0,1)\}) = \mu(\{(0,1,1)\}) = \frac{1}{4}$$

$$\mu(\{(1,0,0)\}) = \mu(\{(0,1,0)\}) = \mu(\{(0,0,1)\}) = \mu(\{(1,1,1)\}) = 0$$

Let f, g, and h be the random variables defined on X as follows:

$$f((j,k,\ell)) = j, \;\; g((j,k,\ell)) = k, \;\; h((j,k,\ell)) = \ell \qquad ((j,k,\ell)\in X)$$

Describe:

$$f_*(\mu),\; g_*(\mu),\; h_*(\mu)$$

and:

$$(f\times g)_*(\mu),\; (f\times h)_*(\mu),\; (g\times h)_*(\mu)$$

Verify that f and g are independent, that f and h are independent, that g and h are independent, but that f, g, and h are not independent.