MATHEMATICS 321
ASSIGNMENT 7
Due: October 28, 2015

Let $\bar{\mathbf{R}}_0^+$ be the set composed of all numbers x for which $0 \leq x$ together with the symbol ∞. Consequently:

$$\bar{\mathbf{R}}_0^+ \equiv \mathbf{R}_0^+ \cup \{\infty\}$$

We supply $\bar{\mathbf{R}}_0^+$ with the operations of addition and multiplication, as usual, but we augment the operations by the following conventions:

$$x + \infty = \infty + x = \infty, \ \ 0 < x \longrightarrow x \times \infty = \infty \times x = \infty$$

$$\infty + \infty = \infty, \ \ \infty \times \infty = \infty, \ \ 0 \times \infty = \infty \times 0 = 0$$

We supply $\bar{\mathbf{R}}_0^+$ with the relation of order, as usual, but we augment the relation by the following convention:

$$x < \infty$$

Let H be the bijective mapping carrying $\bar{\mathbf{R}}_0^+$ to $[0, 1]$, defined as follows:

$$H(\xi) = \frac{\xi}{1 + \xi}$$

We intend that $H(\infty) = 1$. We supply $\bar{\mathbf{R}}_0^+$ with a metric as follows:

$$\delta(\xi_1, \xi_2) \equiv |H(\xi_1) - H(\xi_2)|$$

Note that, for the metric just described, $\bar{\mathbf{R}}_0^+$ is compact. Let:

$$\sigma : \quad \sigma_1 \leq \sigma_2 \leq \ \cdots \ \leq \sigma_j \leq \ \cdots$$

be an increasing sequence in $\bar{\mathbf{R}}_0^+$. Note that σ is convergent.

Let X be a set. Let $\mathcal{A}$ be a family of subsets of X. We say that $\mathcal{A}$ is *borel algebra* iff:

(1) $\emptyset \in \mathcal{A}$
(2) for each subset A of X, if $A \in \mathcal{A}$ then $X \backslash A \in \mathcal{A}$
(3) for any countable family $\mathcal{C}$ of subsets of X, if $\mathcal{C} \subseteq \mathcal{A}$ then $\cup\mathcal{C} \in \mathcal{A}$

We refer to the set X, supplied with a borel algebra $\mathcal{A}$ of subsets of X, as a *borel space*. We refer to the sets in $\mathcal{A}$ as *borel subsets* of X.

Let X be a set. Let $\mathcal{G}$ be a family of subsets of X. Let $\mathbf{A}$ be the collection of all borel algebras of subsets of X which include $\mathcal{G}$. Note that $\mathbf{A} \neq \emptyset$, because the family $\mathcal{X}$ consisting of all subsets of X is contained in $\mathbf{A}$. Let:

$$\mathcal{A} = \bigcap \mathbf{A}$$

Obviously, $\mathcal{A}$ is a borel algebra of subsets of X. Moreover, for any borel algebra $\mathcal{B}$ of subsets of X, if $\mathcal{G} \subseteq \mathcal{B}$ then $\mathcal{A} \subseteq \mathcal{B}$. We say that $\mathcal{A}$ is the *smallest* among all borel algebras of subsets of X which include $\mathcal{G}$. We say that $\mathcal{G}$ *generates* $\mathcal{A}$.

Let X be a metric space, with metric d. Let $\mathcal{T}$ be the family of all open subsets of X. We refer to $\mathcal{T}$ as the *topology* on X. Let $\mathcal{A}$ be the borel algebra of subsets of X generated by the topology $\mathcal{T}$. We regard $\mathcal{A}$ as the *standard* borel algebra of subsets of X, defined relative to the metric d on X. Given a metric space X, we supply X with the standard borel algebra $\mathcal{A}$, without comment.

Let X_1 and X_2 be borel spaces, supplied with borel algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Let F be a mapping carrying X_1 to X_2. We say that F is a *borel mapping* iff, for each borel subset B of X_2, $F^{-1}(B)$ is a borel subset of X_1.

01$^\bullet$ Let X_1 be a borel space, with borel algebra $\mathcal{A}_1$, and let X_2 be a metric space, with metric d_2. Let F be a mapping carrying X_1 to X_2. Assume that, for each open subset V of X_2, $F^{-1}(V)$ is a borel subset of X_1. [The assumption would be true if X_1 were a metric space, with metric d_1, and if F were continuous. Why?] Show that F is a borel mapping.

[HINT. Introduce the family $\mathcal{B}$ consisting of all subsets B of X_2 such that $F^{-1}(B)$ lies in $\mathcal{A}_1$. Show that $\mathcal{B}$ is a borel algebra of subsets of X_2. By hypothesis, the topology on X_2 is a subfamily of $\mathcal{B}$. Draw the conclusion.]

02$^\bullet$ Let X be a borel space, with borel algebra $\mathcal{A}$. Let:

$$f_1, f_2, \ \ldots, \ f_j, \ \ldots$$

be a sequence of functions defined on X with values in $\mathbf{R}$. Let the sequence be pointwise convergent and let g be the corresponding pointwise limit:

$$g(x) = \lim_{j \to \infty} f_j(x)$$

where x is any member of X. Assume that, for each index j, f_j is borel. Show that g is borel.

[HINT. Of course, we assume that $\mathbf{R}$ is supplied with the standard borel algebra. Verify that, for each rational number r and for each x in X, $r < g(x)$ iff there is some rational number s such that $r < s$ and such that the sequence:

$$f_1(x), f_2(x), \ \ldots \ , f_j(x), \ \ldots$$

is eventually in $(s, \longrightarrow)$. That is:

$$g^{-1}((r, \longrightarrow)) = \bigcup_{r<s} \bigcup_{k=1}^{\infty} \bigcap_{\ell=k}^{\infty} f_\ell^{-1}((s, \longrightarrow))$$

Draw the conclusion.]