MATHEMATICS 321
ASSIGNMENT 6
Due: October 14, 2015

01° Let **D** be the unit disk in **C**, consisting of all complex numbers ζ such that $|\zeta| \leq 1$. Let $\mathbf{A}'$ be the subalgebra of $\mathbf{C}(\mathbf{D})$ consisting of all functions of the form:

$$f(\zeta) = \sum_{j=0}^{n} \alpha_j \zeta^j \qquad\qquad (\zeta \in \mathbf{D})$$

where n is any nonnegative integer and where the various α_j are any complex numbers. Let $\mathbf{A}''$ be the subalgebra of $\mathbf{C}(\mathbf{D})$ consisting of all functions of the form:

$$g(\zeta) = \sum_{j=0}^{n} \sum_{k=0}^{n} \beta_{jk} \zeta^j \zeta^{*k} \qquad\qquad (\zeta \in \mathbf{D})$$

where n is any nonnegative integer and where the various β_{jk} are any complex numbers. Verify that both $\mathbf{A}'$ and $\mathbf{A}''$ separate points in **D**. Show that $\mathbf{A}''$ is involutory while $\mathbf{A}'$ is not. Prove that $\mathbf{A}''$ is dense in $\mathbf{C}(\mathbf{D})$ while $\mathbf{A}'$ is not.

02° Let X_1 and X_2 be compact spaces. Let f be a complex valued function defined and continuous on $X_1 \times X_2$. Let r be any positive number. Show that there are a nonnegative integer ℓ, complex valued functions:

$$g_j \qquad (0 \leq j \leq \ell)$$

defined and continuous on X_1, and complex valued functions:

$$h_j \qquad (0 \leq j \leq \ell)$$

defined and continuous on X_2 such that, for each (ξ, η) in $X_1 \times X_2$:

$$\Big| f(\xi, \eta) - \sum_{j=0}^{\ell} g_j(\xi) h_j(\eta) \Big| \leq r$$

03° Let **T** be the unit circle in **C** consisting of all complex numbers τ such that $|\tau| = 1$. Let **A** be the subalgebra of $\mathbf{C}(\mathbf{T})$ consisting of all functions of the form:

$$f(\tau) = \sum_{j=-n}^{n} \alpha_j \tau^j = \sum_{j=-n}^{n} \alpha_j e^{ij\theta} \qquad (\tau = e^{i\theta})$$

where n is any nonnegative integer and where the various α_j are any complex numbers. Verify that **A** separates points in **T**. Show that **A** is involutory. Conclude that **A** is dense in $\mathbf{C}(\mathbf{T})$.

04• With reference to the foregoing example, we describe an interesting fact. Let f be any function in $\mathbf{C}(\mathbf{T})$. We form the Fourier Coefficients of f as follows:

$$\gamma_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-ij\theta}d\theta$$

We form the corresponding Fourier Sequences:

$$s_n(e^{i\theta}) = \sum_{j=-n}^{n} \gamma_j e^{ij\theta}, \quad \sigma_m(e^{i\theta}) = \frac{1}{m}\sum_{n=0}^{m-1} s_n(e^{i\theta})$$

In general, s represents f rather loosely. It converges to f in the Integral Metric, a rather weak condition. However, σ converges to f uniformly.