MATHEMATICS 212
ASSIGNMENT 8
Due: April 8, 2015

01° Let I be the 1-cube in $\mathbf{R}$ defined as follows:

$$t \in I \quad \text{iff} \quad 0 \leq t \leq 1$$

Let C be the simple 1-chain in $\mathbf{R}^3$ defined as follows:

$$C(t) = (x, y, z) = (cos(4\pi t), sin(4\pi t), 4\pi t) \qquad (t \in I)$$

Let λ be the 1-form on $\mathbf{R}^3$ defined as follows:

$$\lambda = -y\,dx + x\,dy + dz$$

Compute:

$$\int_C \lambda = \int_I H^*(y\,dx - x\,dy + dz) = \int_0^1 f(t)dt$$

Of course, you must first compute f.

02° Let I^2 be the 2-cube in $\mathbf{R}^2$ defined as follows:

$$(u, v) \in I^2 \quad \text{iff} \quad 0 \leq u \leq 1 \;\text{ and }\; 0 \leq v \leq 1$$

Let H be the simple 2-chain in $\mathbf{R}^3$ defined as follows:

$$H(u, v) = (x, y, z) = (u + v, u^2, v^2) \qquad ((u, v) \in I^2)$$

Let λ be the 2-form on $\mathbf{R}^3$ defined as follows:

$$\lambda = dxdy + y\,dxdz$$

Compute:

$$\int_H \lambda = \int_{I^2} H^*(dxdy + y\,dxdz) = \int_0^1 \int_0^1 g(u, v)dudv$$

Of course, you must first compute g.

03° Let V be a vector field defined on (a subset of) $\mathbf{R}^3$:

$$V = (A, B, C)$$

Let λ be the corresponding differential 1-form:

$$\lambda = Adx + Bdy + Cdz$$

In turn, let I^2 be the 2-cube in $\mathbf{R}^2$ and let H be the simple 2-chain in $\mathbf{R}^3$ defined as follows:

$$H(u, v) = (x, y, z) = (a(u, v), b(u, v), c(u, v)) \qquad ((u, v) \in I^2)$$

Let the range $S = H(I^2)$ of H be included in the domain of V. Of course, there must be a function h defined on I^2 such that:

$$H^*(*\lambda) = h\, dudv$$

Show that:

$$h = \hat{V} \bullet (P \times Q) = (\hat{V} \bullet N)\|P \times Q\|$$

where $\hat{V} = V \cdot H$, where P and Q are the columns of the total derivative DH of H:

$$P = \begin{pmatrix} \partial x/\partial u \\ \partial y/\partial u \\ \partial z/\partial u \end{pmatrix}, \qquad Q = \begin{pmatrix} \partial x/\partial v \\ \partial y/\partial v \\ \partial z/\partial v \end{pmatrix}$$

and where:

$$N = \frac{1}{\|P \times Q\|}(P \times Q)$$

Conclude that:

$$\begin{aligned} \int_H *\lambda &= \int_0^1 \int_0^1 \big(V(H(u, v)) \bullet N(u, v)\big) \|(P \times Q)(u, v)\| dudv \\ &= \int_0^1 \int_0^1 V(H(u, v)) \bullet (P \times Q)(u, v) dudv \end{aligned}$$

In this way, we see that the formalism of differential forms reproduces the classical definition of surface integrals.

04° In context of the preceding problem, consider the case in which the restriction of V to S "coincides" with N. More precisely:

$$\hat{V} = N$$

Show that, in such a case:

$$\int_H *\lambda = \sigma$$

where σ is the surface area of S. In such a case, we interpret $*\lambda$ to be the *surface area* 2-form for S. For example, let r be any positive number and let H be the simple 2-chain in $\mathbf{R}^3$ defined as follows:

$$\begin{aligned} H(u,v) &= (x,y,z) \\ &= (rcos(\frac{\pi}{2}\bar{v})cos(\pi\bar{u}), rcos(\frac{\pi}{2}\bar{v})sin(\pi\bar{u}), rsin(\frac{\pi}{2}\bar{v}))) \end{aligned} \qquad ((u,v) \in I^2)$$

where:

$$\bar{u} = 2u - 1, \quad \bar{v} = 2v - 1$$

Of course, $S = H(I^2)$ is the sphere in $\mathbf{R}^3$ for which the radius is r. Describe a vector field V for which $N = V \cdot H$ and calculate $*\lambda$.

05$^\bullet$ For class discussion. Let ρ and σ be real numbers for which $0 < \rho < \sigma$. Let I^3 be the 3-cube in $\mathbf{R}^3$ and let H be the simple 3-chain in $\mathbf{R}^3$ defined as follows:

$$H: \quad \begin{pmatrix} u \\ v \\ w \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \big(\sigma + \rho u\, cos(2\pi w)\big) cos(2\pi v) \\ \big(\sigma + \rho u\, cos(2\pi w)\big) sin(2\pi v) \\ \rho u\, sin(2\pi w) \end{pmatrix}$$

where:

$$0 \leq u \leq 1, \quad 0 \leq v \leq 1, \quad 0 \leq w \leq 1$$

Let $T = H(I^3)$ be the range of H. Clearly, T is a (solid) torus. Describe the boundary δH of H:

$$\delta H = \sum_{\alpha=0}^{1} \sum_{j=1}^{3} (-1)^{\alpha+j} H^{j,\alpha}$$

where $H^{j,\alpha}$ is the simple 2-chain in $\mathbf{R}^n$ defined as follows:

$$\begin{aligned} H^{1,0}(t_1,t_2) &= H(0,t_1,t_2) \\ H^{1,1}(t_1,t_2) &= H(1,t_1,t_2) \\ H^{2,0}(t_1,t_2) &= H(t_1,0,t_2) \\ H^{2,1}(t_1,t_2) &= H(t_1,1,t_2) \\ H^{3,0}(t_1,t_2) &= H(t_1,t_2,0) \\ H^{3,1}(t_1,t_2) &= H(t_1,t_2,1) \end{aligned}$$