MATHEMATICS 212
ASSIGNMENT 6
Due: March 11, 2015

01° Let ρ and σ be real numbers for which $0 < \rho < \sigma$. Let T be the solid torus in $\mathbf{R}^3$ defined as follows:

$$\begin{aligned} x &= (\sigma + r cos(\theta))cos(\phi) \\ y &= (\sigma + r cos(\theta))sin(\phi) \quad (0 \leq r \leq \rho, 0 \leq \phi \leq 2\pi, 0 \leq \theta \leq 2\pi)) \\ z &= r\, sin(\theta) \end{aligned}$$

Let S be the toral surface of T:

$$\begin{aligned} x &= (\sigma + \rho cos(\theta))cos(\phi) \\ y &= (\sigma + \rho cos(\theta))sin(\phi) \quad (0 \leq \phi \leq 2\pi, 0 \leq \theta \leq 2\pi)) \\ z &= \rho\, sin(\theta) \end{aligned}$$

Let F be the vector field defined on $\mathbf{R}^3$ as follows:

$$F(x, y, z) = (x, y, z) \qquad ((x, y, z) \in \mathbf{R}^3)$$

Verify the following instance of Gauss' Theorem:

$$\int\int_S F \bullet dS = \int\int\int_T div(F) dV$$

02° Let ρ be a real number for which $0 < \rho$. Let S be the semi spherical surface in $\mathbf{R}^3$ defined as follows:

$$\begin{aligned} x &= \rho\, cos(\theta)cos(\phi) \\ y &= \rho\, cos(\theta)sin(\phi) \quad (-\pi \leq \phi \leq \pi, 0 \leq \theta \leq \pi/2) \\ z &= \rho\, sin(\theta) \end{aligned}$$

Let Γ be the boundary curve for S:

$$\begin{aligned} x &= \rho\, cos(\omega) \\ y &= \rho\, sin(\omega) \quad (-\pi \leq \omega \leq \pi) \\ z &= 0 \end{aligned}$$

Let F be the vector field defined on $\mathbf{R}^3$ as follows:

$$F(x, y, z) = (-y, x, xyz) \qquad ((x, y, z) \in \mathbf{R}^3)$$

Verify the following instance of Stokes' Theorem:

$$\int_\Gamma F\, ds = \int\int_S (\nabla \times F) \bullet dS$$

03° Let H be the stereographic parametrization of $\mathbf{S}^2$, defined as follows:

$$H(u,v) = (x,y,z)$$

where (u,v) is any ordered pair in $\mathbf{R}^2$ and where (x,y,z) is the corresponding ordered triple in $\mathbf{R}^3$:

$$x = \frac{2u}{u^2+v^2+1}, \quad y = \frac{2v}{u^2+v^2+1}, \quad z = \frac{u^2+v^2-1}{u^2+v^2+1}$$

One can easily verify that H carries $\mathbf{R}^2$ bijectively to $\mathbf{S}^2\backslash\{N\}$, where N is the "north pole" in $\mathbf{S}^2$:

$$N = (0,0,1)$$

Show that H is a conformal parametrization of $\mathbf{S}^2$. That is, show that, for any ordered pair (u,v) in $\mathbf{R}^2$, the columns of $DH(u,v)$ are orthogonal and they have the same length:

$$DH(u,v) = (\, P(u,v) \quad Q(u,v)\,) = \begin{pmatrix} x_u(u,v) & x_v(u,v) \\ y_u(u,v) & y_v(u,v) \\ z_u(u,v) & z_v(u,v) \end{pmatrix}$$

Apply the parametrization H of $\mathbf{S}^2$ to compute the surface area of $\mathbf{S}^2$. You should find 4π, of course.