MATHEMATICS 211
ASSIGNMENT 11
Due: December 3, 2014

01• Memorize the Greek alphabet:

α	alpha	A
β	beta	B
γ	gamma	Γ
δ	delta	Δ
ϵ	epsilon	E
ζ	zeta	Z
η	eta	H
θ	theta	Θ
ι	iota	I
κ	kappa	K
λ	lambda	Λ
μ	mu	M
ν	nu	N
ξ	xi	Ξ
o	omicron	O
π	pi	Π
ρ	rho	P
σ	sigma	Σ
τ	tau	T
υ	upsilon	Υ
ϕ	phi	Φ
χ	chi	X
ψ	psi	Ψ
ω	omega	Ω

02° Let ϕ be a scalar field and let F be a vector field on $\mathbf{R}^3$. By definition, ϕ is a function for which the domain is (a suitable subset of) $\mathbf{R}^3$ and the codomain is $\mathbf{R}$:

$$\phi(x, y, z)$$

while F is a function for which the domain is (a suitable subset of) $\mathbf{R}^3$ and the codomain is $\mathbf{R}^3$:

$$F(x, y, z) = (u(x, y, z),\ v(x, y, z),\ w(x, y, z))$$

where u, v, and w are (in effect) scalar fields, the components of F.

One defines the following operators acting on ϕ and F:

(•) **Gradient**

$\nabla\phi = (\partial\phi/\partial x,\, \partial\phi/\partial y,\, \partial\phi/\partial z)$

(•) **Curl**

$\nabla\times F = (\partial w/\partial y - \partial v/\partial z,\, \partial u/\partial z - \partial w/\partial x,\, \partial v/\partial x - \partial u/\partial y)$

(•) **Divergence**

$\nabla \bullet F = \partial u/\partial x + \partial v/\partial y + \partial w/\partial z$

(•) **Laplacian**

$\nabla^2\phi = \nabla \bullet (\nabla\phi) = \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + \partial^2\phi/\partial z^2$

Show that:

$$\nabla\times(\nabla\phi) = (0,0,0) \quad \text{and} \quad \nabla \bullet (\nabla\times F) = 0$$

[Proceed by straightforward computation.]

03° Given vector fields G and H on $\mathbf{R}^3$, show that:

$$\nabla \bullet (G\times H) = (\nabla\times G) \bullet H - G \bullet (\nabla\times H)$$

[Proceed by straightforward computation.]

04° Given a vector field F on $\mathbf{R}^3$, show that:

$$\nabla\times(\nabla\times F) = \nabla(\nabla \bullet F) - \nabla^2 F$$

Of course, ∇^2 acts on F component by component.

[Proceed by straightforward computation.]

05° Let ϕ be the scalar field on $\mathbf{R}$ defined as follows:

$$\phi(x,y,z) = -\frac{1}{r} \qquad (0 < r)$$

where:

$$r = \sqrt{x^2 + y^2 + z^2}$$

Calculate:

$$-(\nabla\phi)(x,y,z)$$

To that end, note that $\partial r/\partial x = x/r$, $\partial r/\partial y = y/r$, and $\partial r/\partial z = z/r$.

[Apply the Chain Rule. For instance:

$$\frac{\partial\phi}{\partial x}(x,y,z) = \frac{d}{dr}(-\frac{1}{r})(r)\,\frac{\partial r}{\partial x}(x,y,z) = r^{-2}\,\frac{x}{r} = \frac{x}{r^3}$$

Hence:

$$-(\nabla(-\frac{1}{r})(x,y,z) = -(\frac{x}{r^3},\frac{y}{r^3},\frac{z}{r^3})$$

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06° Let D be a subset of $\mathbf{R}^3$ and let F be a vector field on $\mathbf{R}^3$ defined on D:

$$F(x,y,z) = (u(x,y,z),\, v(x,y,z),\, w(x,y,z)) \qquad ((x,y,z)\in D)$$

Let J be a closed finite interval in $\mathbf{R}$:

$$J = [a,b] \qquad (a < b)$$

and let Γ be a (parametrized) curve in $\mathbf{R}^3$ defined on J:

$$\Gamma(t) = (x(t),y(t),z(t)) \qquad (a \leq t \leq b)$$

Let the range of Γ be included in the domain of F:

$$\Gamma(t)\in D \qquad (a\leq t\leq b)$$

In this context, one defines the *line integral* of F over Γ as follows:

$$\begin{aligned}\int_\Gamma F &:= \int_a^b F(\Gamma(t))\bullet\Gamma^\circ(t)dt\\ &= \int_a^b \big[\big(u(x(t),y(t),z(t)\big)x^\circ(t) + \big(v(x(t),y(t),z(t)\big)y^\circ(t) + \big(w(x(t),y(t),z(t)\big)z^\circ(t)\big]dt\end{aligned}$$

For the following particular cases:

$$F(x,y,z) = (-\frac{y}{x^2+y^2},\frac{x}{x^2+y^2},z) \qquad (0 < x^2+y^2)$$

and:

$$\Gamma_1(t) = (cos(t),sin(t),t),\quad \Gamma_2(t) = (cos(t),-sin(t),t) \qquad (0\leq t\leq\pi)$$

show that:

$$(\nabla \times F)(x, y, z) = (0, 0, 0)$$

and:

$$\int_{\Gamma_1} F \neq \int_{\Gamma_2} F$$

Note, however, that the initial points of Γ_1 and Γ_2 coincide. The same is true of the terminal points. Finally, with reference to the preceding problem, replace F by:

$$F(x, y, z) = -(\nabla\phi)(x, y, z)$$

Show that, in this case:

$$\int_{\Gamma_1} F = \int_{\Gamma_2} F$$

[Proceed by straightforward computation. Use the fact that if F is the gradient of a potential then line integrals are "independent of path."]

07° Let α be a function defined on the interval $\mathbf{R}^+ := (0, \infty)$ in $\mathbf{R}$ and let ϕ be the scalar field defined on the region $D := \mathbf{R}^3 \backslash \{(0, 0, 0)\}$ in $\mathbf{R}^3$, as follows:

$$\phi(x, y, z) = \alpha(r) \qquad (0 < r := \sqrt{x^2 + y^2 + z^2})$$

Show that:

$$(\nabla\phi)(x, y, z) = \alpha^\circ(r)\frac{1}{r}(x, y, z) \qquad (0 < r := \sqrt{x^2 + y^2 + z^2})$$

[Proceed by straightforward computation. Apply the chain rule, as in problem 5°. Note that (x, y, z) stands for the value of the identity (!) vector field I at (x, y, z):

$$I(x, y, z) = (x, y, z)$$

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08° Let β be a function defined on the interval $\mathbf{R}^+ := (0, \infty)$ in $\mathbf{R}$ and let F be the vector field defined on the region $D := \mathbf{R}^3 \backslash \{(0, 0, 0)\}$ in $\mathbf{R}^3$, as follows:

$$F(x, y, z) = \beta(r)(x, y, z) \qquad (0 < r := \sqrt{x^2 + y^2 + z^2})$$

Show that:

$$(\nabla \times F)(x, y, z) = (0, 0, 0) \qquad (0 < r := \sqrt{x^2 + y^2 + z^2})$$

Find a scalar field ϕ for which:

$$F(x, y, z) = (\nabla\phi)(x, y, z) \qquad (0 < r := \sqrt{x^2 + y^2 + z^2})$$

To do so, apply the foregoing problem. Work out the details for the following case:

$$\beta(r) = r^a \qquad (0 < r)$$

where a is any real number.

[By the foregoing problem, you need only find α such that:

$$\alpha^\circ(r) = r\beta(r)$$

For the special case, that would be simple.]

09° Let J be an interval in $\mathbf{R}$ and let D be a region in $\mathbf{R}^3$. Let ϕ be a scalar field "defined on D but depending on t":

$$\phi(t, x, y, z) \qquad (t \in J, \;\; (x, y, z) \in D)$$

One defines the following operator acting on ϕ:

(•) **d'Alembertian**

$$\Box^2\phi = \partial^2\phi/\partial t^2 - \nabla^2\phi = \partial^2\phi/\partial t^2 - \partial^2\phi/\partial x^2 - \partial^2\phi/\partial y^2 - \partial^2\phi/\partial z^2$$

Let G and H be vector fields "defined on D but depending on t":

$$G(t, x, y, z), \quad H(t, x, y, z) \qquad (t \in J, \;\; (x, y, z) \in D)$$

and satisfying the following relations:

$$\nabla \bullet G = 0, \quad \nabla \bullet H = 0$$

$$\partial G/\partial t - \nabla \times H = (0, 0, 0), \qquad \partial H/\partial t + \nabla \times G = (0, 0, 0)$$

Show that:

$$\Box^2 G = (0, 0, 0) \quad \text{and} \quad \Box^2 H = (0, 0, 0)$$

Of course, $\partial/\partial t$ and $\Box^2$ act on G and H component by component. Conclude that any one of the components of G and H, let it be ϕ, satisfies the **wave equation**:

$$\Box^2\phi = 0$$

[See problem 4°. The computation should be brief. You do not need to introduce the components of G and H.]