MATHEMATICS 211
ASSIGNMENT 5
Due: October 8, 2014

01° Let A be an antisymmetric matrix:

$$A = \begin{pmatrix} 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \end{pmatrix}$$

Compute the determinant of A.

[We have:

$$det(A) = 0.0.0 - 0.u.v - w.(-w).0 + w.u.v + (-v).(-w).(-u) - (-v).0.v = 0$$

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02° Let a be a real number, distinct from 0. Let f be the mapping carrying $\mathbf{R}^3\backslash\{0\}$ to $\mathbf{R} = \mathbf{R}^1$, defined as follows:

$$f(x, y, z) := (x^2 + y^2 + z^2)^a$$

Find the value(s) of a for which:

$$f_{xx}(x, y, z) + f_{yy}(x, y, z) + f_{zz}(x, y, z) = 0$$

[Let us introduce the notation:

$$r^2 = x^2 + y^2 + z^2 \quad \text{so} \quad f(x, y, z) = r^{2a} \ \text{ and } \ r_x = \frac{x}{r},\ r_y = \frac{y}{r},\ r_z = \frac{z}{r}$$

Clearly:

$$f_x = 2ar^{2(a-1)}x, \quad f_{xx} = 2ar^{2(a-1)} + 4a(a-1)r^{2(a-2)}x^2$$

By similar computations for y and z, we find that:

$$f_{xx} + f_{yy} + f_{zz} = 6ar^{2(a-1)} + 4a(a-1)r^{2(a-1)} = 2a(2a+1)r^{2(a-1)}$$

We conclude that:

$$a = 0 \ \text{ or } \ a = -\frac{1}{2}$$

The latter case is the celebrated potential function in celestial mechanics.]

03° Consider the following curve in $\mathbf{R}^3$:

$$\Gamma(t) := (exp(t)cos(t), exp(t)sin(t), t)$$

where t is any real number (soit *time*). Find the angle between the position vector $\Gamma(t)$ and the velocity vector $\Gamma'(t)$ at time $t = \pi/4$.

[This problem involves just a routine computation with classical functions.]

04° Let f be the real valued function defined on $\mathbf{R}^3$ as follows:

$$f(x, y, z) = z - (x^2 + y^2) \qquad ((x, y, z) \in \mathbf{R}^3)$$

Let M be the level set in $\mathbf{R}^3$ defined by the relation:

$$f(x, y, z) = 0$$

Clearly, the (position) vector $(1, 2, 5)$ lies in M. The tangent plane $T_{(1,2,5)}(M)$ to M at $(1, 2, 5)$ consists of the vectors (u, v, w) in $\mathbf{R}^3$ which meet the condition:

$$(f_x(1, 2, 5), f_y(1, 2, 5), f_z(1, 2, 5)) \bullet (u, v, w) = d$$

where d is a suitable number. Find d.

[Clearly:

$$f_x(x, y, z) = -2x, \quad f_y(x, y, z) = -2y, \quad f_z(x, y, z) = 1$$

Of course, $(1, 2, 5)$ must lie on the tangent plane. Hence:

$$d = (f_x(1, 2, 5), f_y(1, 2, 5), f_z(1, 2, 5)) \bullet (1.2.5) = (-2).1 + (-4).2 + 1.5 = -5$$

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05° Let F be the mapping carrying $\mathbf{R}^2$ to $\mathbf{R}^3$ defined by the following relations:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = F(\begin{pmatrix} u \\ v \end{pmatrix}) : \quad \begin{array}{l} x = u \\ y = v \\ z = u^2 + v^2 \end{array}$$

Let M be the range of F. Describe the tangent plane:

$$T_P(M)$$

to M at the point:

$$P = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = F(\begin{pmatrix} 1 \\ 2 \end{pmatrix})$$

By definition, the vectors in $T_P(M)$ have the form:

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + \begin{pmatrix} \circ & \bullet \\ \circ & \bullet \\ \circ & \bullet \end{pmatrix} \begin{pmatrix} u-1 \\ v-2 \end{pmatrix}$$

where:

$$\begin{pmatrix} \circ & \bullet \\ \circ & \bullet \\ \circ & \bullet \end{pmatrix} = DF(\begin{pmatrix} 1 \\ 2 \end{pmatrix})$$

and where:

$$\begin{pmatrix} u \\ v \end{pmatrix}$$

runs through all vectors in $\mathbf{R}^2$. Find a vector:

$$N = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

which is perpendicular to $T_P(M)$. In fact, you can take N to be:

$$\begin{pmatrix} \circ \\ \circ \\ \circ \end{pmatrix} \times \begin{pmatrix} \bullet \\ \bullet \\ \bullet \end{pmatrix}$$

Draw a diagram to illustrate the sense of this problem.

[Of course:

$$DF(\begin{pmatrix} u \\ v \end{pmatrix}) = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2u & 2v \end{pmatrix}$$

Hence:

$$DF(\begin{pmatrix} 1 \\ 2 \end{pmatrix}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 4 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} -2 \\ -4 \\ 1 \end{pmatrix}$$

Now the vectors in the tangent plane have the form:

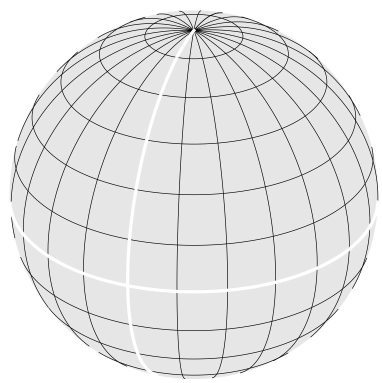
$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} u-1 \\ v-2 \end{pmatrix} = \begin{pmatrix} u \\ v \\ 2u+4v-5 \end{pmatrix}$$

where u and v are any numbers.

06° Let H be the Hipparchus Map:

$$H\begin{pmatrix}\phi\\ \theta\end{pmatrix} = \begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}cos\theta cos\phi\\ cos\theta sin\phi\\ sin\theta\end{pmatrix}$$

where ϕ is the longitude and θ is the latitude. In the following picture of the range $\mathbf{S}^2$ of H:



plot the vector:

$$V = H\begin{pmatrix}\pi/4\\ \pi/6\end{pmatrix} + DH\begin{pmatrix}\pi/4\\ \pi/6\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}$$

Be ye exact.

[The computations are routine. For a nautical perspective, one may ask: What is the *bearing* of V?]

07° Let a and b be any numbers for which $0 < b < a$. Let c be the positive number which satisfies the relation: $b^2 + c^2 = a^2$. Let f be the function defined on $\mathbf{R}^2$ as follows:

$$f(x, y) = (\frac{x}{a})^2 + (\frac{y}{b})^2$$

Let (p, q) be any member of $\mathbf{R}^2$ for which $f(p, q) = 1$. [The set of all such members (p, q) compose an *ellipse* in $\mathbf{R}^2$. We will make a drawing of it in the lectures.] Let α be the angle between the vectors:

$$(p, q) - (-c, 0) \quad \text{and} \quad (f_x(p, q), f_y(p, q))$$

and let β be the angle between the vectors:

$$(p, q) - (+c, 0) \quad \text{and} \quad (f_x(p, q), f_y(p, q))$$

Show that $\alpha = \beta$. This result explains the phenomenon of the Whispering Gallery.

[The problem reduces to verifying the following relation:

$$(1 + \frac{pc}{a^2})^2((p-c)^2 + q^2) = (1 - \frac{pc}{a^2})^2((p+c)^2 + q^2)$$

where:

$$a^2 = b^2 + c^2, \ \ (\frac{p}{a})^2 + (\frac{q}{b})^2 = 1$$

One may proceed by grouping the terms in a savvy manner, or by patiently calculating term by term.]

08$^\bullet$ Find the equation of the tangent plane for the surface:

$$S: \qquad \sqrt{x} + \sqrt{y} + \sqrt{z} = 4 \qquad (0 < x, 0 < y, 0 < z)$$

at the point $(1, 4, 1)$.