MATHEMATICS 211
ASSIGNMENT 1: SOLUTIONS
Due: September 10, 2014

$01^\bullet$ Memorize the Greek alphabet:

α	alpha	A
β	beta	B
γ	gamma	Γ
δ	delta	Δ
ϵ	epsilon	E
ζ	zeta	Z
η	eta	H
θ	theta	Θ
ι	iota	I
κ	kappa	K
λ	lambda	Λ
μ	mu	M
ν	nu	N
ξ	xi	Ξ
o	omicron	O
π	pi	Π
ρ	rho	P
σ	sigma	Σ
τ	tau	T
υ	upsilon	Υ
ϕ	phi	Φ
χ	chi	X
ψ	psi	Ψ
ω	omega	Ω

[I testify that I have memorized the Greek alphabet.]

02° Show that, for any (position) vectors x' and x' in $\mathbf{R}^n$:

$$\langle\!\langle\, x', x'' \,\rangle\!\rangle = \frac{1}{4}(\|x' + x''\|^2 - \|x' - x''\|^2)$$

One refers to this relation as the *Polarization Identity*.

[We have:

$$\begin{aligned} &\frac{1}{4}(\|x' + x''\|^2 - \|x' - x''\|^2) \\ &\qquad = \frac{1}{4}(\langle\!\langle\, x' + x'', x' + x'' \,\rangle\!\rangle - \langle\!\langle\, x' - x'', x' - x'' \,\rangle\!\rangle) \\ &\qquad = \langle\!\langle\, x', x'' \,\rangle\!\rangle \end{aligned}$$

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03° Let x' and x'' be (position) vectors in $\mathbf{R}^3$:

$$x' = \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix}, \qquad x'' = \begin{pmatrix} u'' \\ v'' \\ w'' \end{pmatrix}$$

and let x be the *cross product* of x' and x'':

$$x = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = x' \times x'' = \begin{pmatrix} v'w'' - w'v'' \\ w'u'' - u'w'' \\ u'v'' - v'u'' \end{pmatrix}$$

Verify that:

$$\langle\!\langle\, x', x' \times x'' \,\rangle\!\rangle = 0 \quad \text{and} \quad \langle\!\langle\, x', x' \times x'' \,\rangle\!\rangle = 0$$

[We have:

$$\langle\!\langle\, x', x' \times x'' \,\rangle\!\rangle = u'(v'w'' - w'v'') + v'(w'u'' - u'w'') + w'(u'v'' - v'u'') = 0$$

The second relation follows from a similar computation.]

04° Find the angle θ between the (position) vectors:

$$x' = \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \quad x'' = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

[We have:

$$cos(\theta) = \frac{\langle\!\langle\, x', x'' \,\rangle\!\rangle}{\|x'\|\|x''\|} = -\frac{1}{2}, \quad \text{hence} \quad \theta = \frac{2}{3}\pi$$

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05° Let P be the plane in $\mathbf{R}^3$ composed of all (position) vectors of the form:

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

where s and t are any real numbers. Find a (nonzero) *direction* vector n:

$$n = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

and a real number d such that, for each (position) vector x in $\mathbf{R}^3$, x lies in P iff:

$$\langle\!\langle\, x, n \,\rangle\!\rangle = d$$

[Let us introduce n as follows:

$$n = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Clearly:

$$\langle\!\langle\, n, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \,\rangle\!\rangle = 0 \quad \text{and} \quad \langle\!\langle\, n, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \,\rangle\!\rangle = 0$$

We want to have:

$$\langle\!\langle\, n, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \,\rangle\!\rangle = d$$

so let us take d to be 2. Now let x be any vector in $\mathbf{R}^3$. If x is in P then there are numbers s and t such that:

$$x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Obviously:

$$\langle\!\langle\, n, x \,\rangle\!\rangle = 2 = d$$

Conversely, let x be any vector in $\mathbf{R}^3$:

$$x = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

If $\langle\!\langle\, n, x \,\rangle\!\rangle = d = 2$ then:

$$\langle\!\langle\, n, (x - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}) \,\rangle\!\rangle = 0$$

so that:

$$(u - 1) - v + (w - 1) = 0$$

It follows that:

$$x - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = (u-1) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (w-1) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Consequently, x is in P.]

06$^{\bullet}$ Let a be a positive real number. Let d be the position vector:

$$d = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$$

Let x be any position vector:

$$x = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

For each real number t, one may form the corresponding position vector s on the line joining d and x:

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = s = (1-t)d + tx = \begin{pmatrix} (1-t)a \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} tu \\ tv \\ tw \end{pmatrix}$$

Find the value of t for which $p = 0$. In turn, solve for q and r in terms of a, u, v, and w. Note that the foregoing computation fails for certain position vectors X. In the lectures, we will show that the foregoing procedure provides an algorithm for **perspective drawing**. Show that lines in the *real world* perpendicular to the *picture plane* appear in the picture plane as lines converging to the *vanishing point*. What can be said of the various other lines in the real world?

[Lectures.]