RANDOM VARIABLES AND PROBABILITY DISTRIBUTIONS

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01° We begin with a *probability space*:

$$(X, \mathcal{A}, \pi)$$

We mean to say that $(X, \mathcal{A})$ is a measurable space and that π is a probability measure on $\mathcal{A}$. In turn, we introduce *random variables* on X. These are the various measurable mappings F carrying X to $\mathbf{R}$:

$$F: \quad X \longrightarrow \mathbf{R}$$

The condition that F be measurable means that, for each measurable subset B of $\mathbf{R}$, $F^{-1}(B)$ is a measurable subset of X.

02° Finally, for each such random variable F, we introduce the corresponding probability measure on $\mathbf{R}$:

$$\mu = F_*(\pi)$$

By definition:

$$\mu(B) = \pi(F^{-1}(B))$$

One refers to μ as the *probability distribution* for F.

03° Without apology, we shall restrict attention to random variables for which the *second moment* is finite:

$$\int_X F^2(x)\pi(dx) = \int_{\mathbf{R}} y^2\mu(dy) < \infty$$

Under this restriction, we obtain two basic characteristics for random variables, the *mean* m and the *variance* s^2:

$$m = \int_X F(x)\pi(dx) = \int_{\mathbf{R}} y\,\mu(dy)$$

$$s^2 = \int_X (F(x) - m)^2\pi(dx) = \int_{\mathbf{R}} (y - m)^2\mu(dy)$$

04° In practice, we will focus attention upon a convenient special case. We mean the case of *standard* random variables: those for which $m = 0$ and $s = 1$.

05° We may say that our framework for study shall be the *real Hilbert space*:

$$\mathbf{L}^2(X, \mathcal{A}, \pi)$$

consisting of all square integrable measurable real valued functions F defined on X.

06° Of course, we may take advantage of the conventional inner product, norm, and metric on $\mathbf{L}^2(X, \mathcal{A}, \pi)$, defined as follows:

$$\langle\!\langle\, F, G \,\rangle\!\rangle = \int_X F(x)G(x)\pi(dx)$$

$$\|H\| = \sqrt{\langle\!\langle\, H, H \,\rangle\!\rangle} = \sqrt{\int_X |H(x)|^2 \pi(dx)}$$

$$d(F, G) = \|F - G\| = \sqrt{\int_X |F(x) - G(x)|^2 \pi(dx)}$$

07° Now we may characterize the standard random variables H by the conditions:

$$\langle\!\langle\, H, \bar{1} \,\rangle\!\rangle = 0, \qquad \|H\| = 1$$

where $\bar{1}$ stands for the constant function $\bar{1}$ in $\mathbf{L}^2(X, \mathcal{A}, \pi)$ for which the constant value is 1.

08° Let us turn our attention to sequences of random variables:

$$F_1, F_2, F_3, \ \ldots\ , F_j, \ \ldots$$

The corresponding probability distributions, means, and variances stand as follows:

$$\mu_1, \mu_2, \mu_3, \ \ldots\ , \mu_j, \ \ldots$$

$$m_1, m_2, m_3, \ \ldots\ , m_j, \ \ldots\ ; \qquad s_1^2, s_2^2, s_3^2, \ \ldots\ , s_j^2, \ \ldots$$

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12° Now we have two basic Objectives. Invoking the Ergodic Theorem, we will prove the Strong Law of Large Numbers. Invoking the Theorem of Levy, we will prove the Central Limit Theorem.

13° For the statements and proofs of the foregoing results, we rquire two useful utilities: the *distribution function* Φ and the *characteristic function* $\hat{\mu}$:

$$\Phi(z) = \mu((-\infty, z]), \qquad \hat{\mu}(t) = \int_{\mathbf{R}} e^{ity} \mu(dy)$$

where z and t are any real numbers. In certain cases, we may also introduce a *density function* f:

$$\mu(B) = \int_B f(y)dy$$

where B is any measurable subset of $\mathbf{R}$. One should note that Φ, $\hat{\mu}$, and f are related as follows:

$$\Phi(x) = \int_{-\infty}^{x} f(y)dy, \qquad \hat{\mu}(t) = \int_{\mathbf{R}} e^{ity} f(y)dy$$