DIFFERENTIAL FORMS ON $\mathbf{R}^3$

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On $\mathbf{R}^3$, forms stand as follows:

0-forms:	$f = f.1$	(which are functions)
1-forms:	$p\,dx + q\,dy + r\,dz$	
2-forms:	$u\,dy \wedge dz + v\,dz \wedge dx + w\,dx \wedge dy$	
3-forms:	$h\,dx \wedge dy \wedge dz$	

where f, p, q, r, u, v, w, and h are any functions of x, y, and z. For a k-form λ, one refers to k as the *degree* of the form. One adds forms in the manner expected but one multiplies them by applying the following relations:

$$d\alpha \wedge d\beta + d\beta \wedge d\alpha = 0$$

where α and β stand for any of x, y, and z. For instance, $dx \wedge dx = 0$ and $dy \wedge dz = -dz \wedge dy$. One computes the *exterior derivative* $d\lambda$ of a form λ as follows:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$d(p\,dx + q\,dy + r\,dz) = dp \wedge dx + dq \wedge dy + dr \wedge dz$$

$$\begin{aligned} d(u\,dy \wedge dz + v\,dz \wedge dx + &w\,dx \wedge dy) \\ &= du \wedge dy \wedge dz + dv \wedge dz \wedge dx + dw \wedge dx \wedge dy \end{aligned}$$

$$d(h\,dx \wedge dy \wedge dz) = dh \wedge dx \wedge dy \wedge dz = 0 \ (!)$$

Note that d increases the degree of λ by 1. One computes the *Hodge $*$-operator* on a form λ by applying the following relations:

$$*1 = dx \wedge dy \wedge dz$$

$$*dx = dy \wedge dz, \quad *dy = dz \wedge dx, \quad *dz = dx \wedge dy$$

$$*(dy \wedge dz) = dx, \quad *(dz \wedge dx) = dy, \quad *(dx \wedge dy) = dz$$

$$*(dx \wedge dy \wedge dz) = 1$$

Consequently, $dy \wedge (*dy) = dx \wedge dy \wedge dz$, $(dz \wedge dx) \wedge (*(dz \wedge dx)) = dx \wedge dy \wedge dz$, and so forth. Moreover:

$$*(p\,dx + q\,dy + r\,dz) = p\,dy \wedge dz + q\,dz \wedge dx + r\,dx \wedge dy$$