Threshold Dynamical Systems

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What is a Threshold Dynamical System?

- Implementation of dynamical systems on graphs.
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- Similar to cellular automata.
Specifics

- The vertices of a given graph $Y$ can either be on or off. This is known as the vertex’s state.
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- Associated with each vertex is also a *threshold*. 
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- The vertices of a given graph $Y$ can either be on or off. This is known as the vertex’s state.
- Associated with each vertex is also a threshold.
- The state of a vertex is governed by a threshold function.
Threshold Functions

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- Let $k$ be the threshold of vertex $v$; then $v$ will turn on if and only if the number of on vertices in $v$’s 1-neighborhood exceeds $k$. 
Example: Simultaneous Update
Use of TDS

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- Social contagions: rumors, smoking
- Biological contagions: malaria, acquired viral load
State Space
What Do We Want To Know?

- Long term behavior of the system: fixed points, limit cycles
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- Long term behavior of the system: fixed points, limit cycles
- States space informs us of overall population ‘infected’
Evolving Thresholds

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• This can represent changing immunity.
• Threshold of $v$ is increased when it is turned on, or its threshold decreased when it is turned off, or both. Thresholds are bounded.
No Limit Cycles

It is already known that non-evolving threshold systems do not have limit cycles. We extended this result to evolving thresholds. This depends on the boundedness of the thresholds.
Fixed Points

The number of fixed points in the state space remains the same among increasing, decreasing, and mixed evolving thresholds and regardless of update order. Notice that once a point in state space is fixed, it remains fixed in all these different systems.
Number of fixed points on a path of length $n$ is $\text{Fib}(3n - 1)$. This is the same for increasing, decreasing, and mixed evolving threshold systems. This was computed through recursion.
Number of fixed points on a circle with $n$ vertices is $2 + \text{Luc}(3n - 1)$. This was computed through the matrix transfer method.
Because the number of vertices does not determine a unique tree, we did not attempt to find a formula based on $n$. Instead, we found an algorithm to compute the number of fixed points in state space by considering the tree as an intersection of paths.
Further Research

There is still much room for future work in the following areas:

• Other types of evolving threshold functions.
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- Other types of evolving threshold functions.
- Graph classes that more accurately represent social networks.
- Stability of limit sets.
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