In Search of Combinatorial Fibonacci Identities

Summer Research 2010

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NUMS, April 9 , 2011

Outline

Fibonacci Sequence

- Defined Recursively:

$$f_0 = 1, \quad f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2}$$

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- $\{1, 1, 2, 3, 5, 8, 13, \dots\}$
- this definition is off by one from the usual definition

Combinatorial Interpretation

Theorem (Benjamin, Quinn)

The number of ways to tile an $(1 \times n)$*-board with* 1×1 *squares and* 1×2 *dominoes is* f_n

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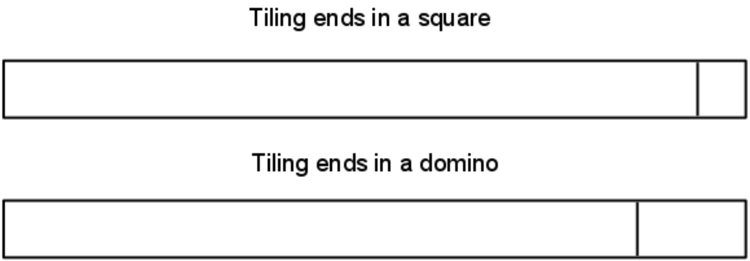

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Theorem

For $n, k \geq 1$, $f_{n+k} = f_n f_k + f_{n-1} f_{k-1}$

Lucas Numbers

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- $\{2, 1, 3, 4, 7, 11, 18, \ldots\}$

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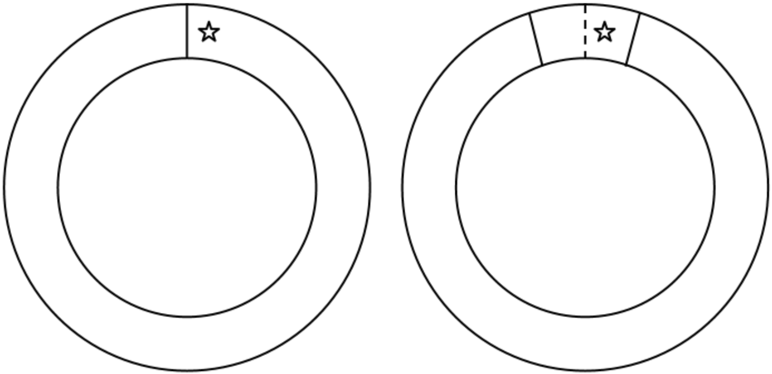
A Useful Identity

Theorem (Benjamin, Quinn)

For $n \geq 2$, $L_n = f_n + f_{n-2}$



Zeckendorf Representations

Theorem (Zeckendorf)

For $n \in \mathbb{N}$ there exists a unique sequence $\{a_j\}_{j=1}^{M} \subseteq \mathbb{N}$ such that $a_{j+1} > a_j + 1$ and

$$n = \sum_{j=1}^{M} f_{a_j}.$$

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We seek Combinatorial Proofs of Zeckendorf Representations. In particular of numbers like $f_n f_k$, $2f_n f_k$, $L_n f_k$, $L_n L_k$.

A Useful Lemma

Lemma

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$$f_{n-2} f_m - f_{n-1} f_{m-1} = (-1)^m f_{n-m}$$

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How many ways can we tile an $(m+n-2)-$board such that there is a fault at $n-2$ but there is not a fault at $n-1$?

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- If m is even, A has one more member than B and vice versa.

A Picture to Convince You

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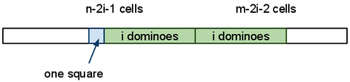
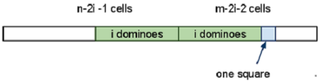
Fault at $n - 2$



Fault at $n - 1$



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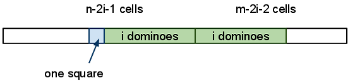
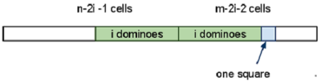
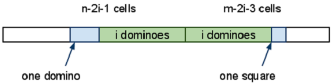


Fault at $n-1$



Fault at $n-2$



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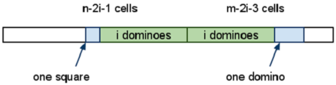

What can we prove with this?

Theorem

For $n - 2 > 2k > 1$

$$L_{2k}L_n = f_{n+2k} + f_{n+2k-2} + f_{n-2k} + f_{n-2k-2}$$

Proof.

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- Use the above lemma twice, let $m \to 2k$ and $m \to 2k, n \to n-2$.

□

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- Thinking about $f_{2k}L_n$ yields ZR for $f_m f_{2k} L_n, f_m L_{2k} L_n, L_m L_{2k} L_n$.

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OPEN PROBLEMS:

- Do the above in a single map.

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- Thinking about $f_{2k}L_n$ yields ZR for $f_mf_{2k}L_n, f_mL_{2k}L_n, L_mL_{2k}L_n$.

OPEN PROBLEMS:

- Do the above in a single map.
- What about $2k + 1$?

Acknowledgments

- Pacific University Mathematics Faculty
- Pacific Research Institute for Science and Mathematics,
- Office of the Dean of Arts and Sciences of Pacific University,
- Involve: Journal of Mathematics,
- Arthur Benjamin and Jennifer Quinn for the *Proofs That Really Count*.