Electromagnetic Modeling and Simulation

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April 7, 2011

Maxwell's Equations

$$\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} = \nabla \times \mathbf{H} \qquad \text{(Ampere)}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \qquad \text{(Faraday)}$$

$$\nabla \cdot \mathbf{D} = \rho \qquad \text{(Poisson)}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(Gauss)}$$

$\mathbf{E} =$	Electric field vector	$\mathbf{D} =$	Electric displacement
$\mathbf{H} =$	Magnetic field vector	$\mathbf{B} =$	Magnetic flux density
$\rho =$	Electric charge density	$\mathbf{J} =$	Current density

These are often referred to as the curl equations.

Constitutive Laws

The Constitutive Laws describe the response of the medium to the electromagnetic field.

$$\mathbf{D} = \varepsilon \mathbf{E} + \mathbf{P}$$
$$\mathbf{B} = \mu \mathbf{H} + \mathbf{M}$$
$$\mathbf{J} = \sigma \mathbf{E} + \mathbf{J}_s$$

$\mathbf{P} =$	Polarization	$\varepsilon =$	Electric permittivity
$\mathbf{M} =$	Magnetization	$\mu =$	Magnetic permeability
$\mathbf{J}_s =$	Source Current	$\sigma =$	Electric conductivity

Evolution in Time

The time evolution of our electromagnetic field in dispersive media is specified by the curl equations

$$\varepsilon_0 \varepsilon_\infty \frac{\partial \mathbf{E}}{\partial t} = -\sigma \mathbf{E} - \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{H}$$

$$\mu_0 \frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E}$$

We assume that there is no magnetization and thus $\mu = \mu_0$. Note for simplicity we neglect the source term, $\mathbf{J}_s$, in our equations. We also note that $\varepsilon = \varepsilon_0 \varepsilon_\infty$ where ε_0 is the permittivity of free space and ε_∞ is the relative permittivity.

Polarization

- From the Constitutive Laws

$$\mathbf{D} = \varepsilon \mathbf{E} + \mathbf{P}$$

  where P is the polarization, ε is the electric permittivity, and D is the electric displacement.
- P can be expressed as the convolution of the Electric field and a dielectric response function(DRF) denoted g. The DRF can be thought of as a memory effect caused by the dielectric.

$$\mathbf{P}(t, \mathbf{x}) = g \star \mathbf{E}(t, \mathbf{x}) = \int_0^t g(t-s, \mathbf{x}; q)\mathbf{E}(s, \mathbf{x})ds,$$

Debye Model with Distributions

For our research we will be using the Debye model, which specifies a Dielectric Response Function of

$$g(t,x) = \frac{\tilde{\varepsilon}_d}{\tau} e^{-\frac{t}{\tau}}$$

where $\tilde{\varepsilon}_d = \varepsilon_0(\varepsilon_s - \varepsilon_\infty)$. The relaxation time, τ, is typically taken to be a scalar, but can be more realistically modeled by a probability distribution. P with this DRF satisfies the ODE:

$$\tau \frac{\partial \mathscr{P}}{\partial t} + \mathscr{P} = \tilde{\varepsilon}_d E$$

This ODE is stochastic if τ is a random variable.

Simplifying to One Spatial Dimension

Assume that the electric field is polarized so that it oscillates in the x-direction and travels in the z-direction. Our equations simplify to:

$$\varepsilon_0 \varepsilon_\infty \frac{\partial E_x}{\partial t} = -\frac{\partial H_y}{\partial z} - \sigma E_x - \frac{\partial P_x}{\partial t}$$

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0} \frac{\partial E_x}{\partial z}$$

From now on, we will let $E := E_x$, $H := H_y$, and $P := P_x$.

Polynomial Chaos: Introduction

- Generalized Polynomial Chaos is a method to describe stochastic (random) solutions to differential equations using a basis of orthogonal polynomials.
- We wish to find a solution to the ordinary differential equation which describes the polarization of a Debye dielectric,

$$\tau \frac{\partial \mathscr{P}}{\partial t} + \mathscr{P} = \tilde{\varepsilon}_d E$$

We can represent the exact solution at each point in space as:

$$\mathscr{P}(\xi) = \sum_{i=0}^{\infty} \alpha_i(t)\phi_i(\xi).$$

Polynomial Chaos: Scaling

The first step is to substitute our solution back into our ODE. We also scale the random variable τ in order to preserve the orthogonality regardless of the domain of the random variable. This produces,

$$(r\xi + m)\sum_{i=0}^{\infty} \dot{\alpha}_i(t)\phi_i(\xi) + \sum_{i=0}^{\infty} \alpha_i(t)\phi_i(\xi) = \tilde{\varepsilon}_d E$$

where m and r are the shift and scaling parameters, respectively. ξ is a random variable with some startard distribution, meaning,

$$\tau = r\xi + m$$

Polynomial Chaos:Recurrence Relation

- All orthogonal polynomials obey the following recurrence relation:

$$\xi\phi_n(\xi) = a_n\phi_{n+1}(\xi) + b_n\phi_n(\xi) + c_n\phi_{n-1}(\xi).$$

- We substitute the recurrence relation into our ODE to remove the explicit dependence on the random variable ξ. Rearranging we obtain,

$$r\sum_{i=0}^{\infty}\dot{\alpha}_i(t)[a_i\phi_{i+1}(\xi)+b_i\phi_i(\xi)+c_i\phi_{i-1}(\xi)] + \sum_{i=0}^{\infty}[\alpha_i(t)+m\dot{\alpha}_i(t)\phi_i(\xi)] = \tilde{\varepsilon}_d E.$$

Polynomial Chaos: Inner Product

- We project onto the finite dimensional random space spanned by $\{\phi_j\}_{j=0}^{p}$ by taking the weighted inner product with each basis function.

$$r\sum_{i=0}^{p}\dot{\alpha}_i(t)\left[a_i\langle\phi_{i+1},\phi_j\rangle_w + b_i\langle\phi_i,\phi_j\rangle_w + c_i\langle\phi_{i-1},\phi_j\rangle_w\right]$$
$$+\sum_{i=0}^{p}[\alpha_i(t)+m\dot{\alpha}_i(t)]\langle\phi_i,\phi_j\rangle_w = \tilde{\varepsilon}_d E\langle\phi_0,\phi_j\rangle_w.$$

- In the above equation, $\langle\phi_i,\phi_j\rangle_w$ is the weighted inner product of ϕ_i and ϕ_j. This is defined as:

$$\langle\phi_i,\phi_j\rangle_w := \int_\Omega \phi_i(\xi)\phi_j(\xi)W(\xi)d\xi = \delta_{ij}\sqrt{h_i h_j}$$

  where δ_{ij} is the Kronecker delta function, $W(\xi)$ is our weighting function, Ω is the domain, and $h_i = \langle\phi_i,\phi_i\rangle_w$.

Polynomial Chaos: System of equations

This produces a system of equations which can be expressed in matrix form as:

$$\mathrm{A}\dot{\vec{\alpha}}(t) + \vec{\alpha}(t) = \vec{f}$$

where $\mathrm{A} = r\mathrm{M} + m\mathrm{I}$ and

$$\mathrm{M} = \begin{pmatrix} b_0 & c_1 & 0 & \cdots & 0 \\ a_0 & b_1 & c_2 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & a_{p-2} & b_{p-1} & c_p \\ 0 & \cdots & 0 & a_{p-1} & b_p \end{pmatrix} \text{ and } \vec{f} = \begin{pmatrix} \tilde{\varepsilon}_d E \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where a_n, b_n and c_n are the coefficients from the recurrence relation.

Popular Distributions

Distribution	Polynomial	Support
Gaussian	Hermite	$(-\infty, \infty)$
gamma	Laguerre	$[0, \infty)$
beta	Jacobi	$[a, b]$
uniform	Legendre	$[a, b]$

The polynomials are orthogonal on the support of the distribution with respect to a weighting function, which is proportional to the density function of the distribution.

Jacobi Polynomials

We are primarily concerned with two types of orthogonal polynomials, Jacobi Polynomials correspond to the beta distribution and have the following recursion coefficients,

$$a_n = \frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}$$
$$b_n = \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$$
$$c_n = \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}$$

These polynomials are orthogonal on the domain $\Omega = [-1,1]$ with respect to the weighting function,

$$w_n(\xi) = (\xi-1)^{\alpha}(1+\xi)^{\beta}.$$

Legendre Polynomials

The Legendre Polynomials are a special case of Jacobi Polynomials are where $\alpha = \beta = 0$. They have the following recursion coefficients

$$a_n = \frac{n+1}{2n+1}$$
$$b_n = 0$$
$$c_n = \frac{n}{2n+1}$$

They are orthogonal over the domain $\Omega = [-1,1]$ with respect to a constant, making them ideally suited for use with a uniform distribution.

Polynomial Chaos Model

We have replaced our original stochastic ODE with the system of deterministic ODEs. Additionally, we substitute the expected value of the stochastic polarization, $\mathbb{E}(\mathscr{P}) \approx \alpha_0$, for the macroscopic polarization, P. Our equations now have the following form:

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu_0}\frac{\partial E}{\partial z}$$

$$\tilde{\varepsilon_\infty}\frac{\partial E}{\partial t} = -\frac{\partial H}{\partial z} - \sigma E - \frac{\partial \alpha_0}{\partial t}$$

$$\mathrm{A}\dot{\vec{\alpha}} + \vec{\alpha} = \vec{f}.$$

Discretization

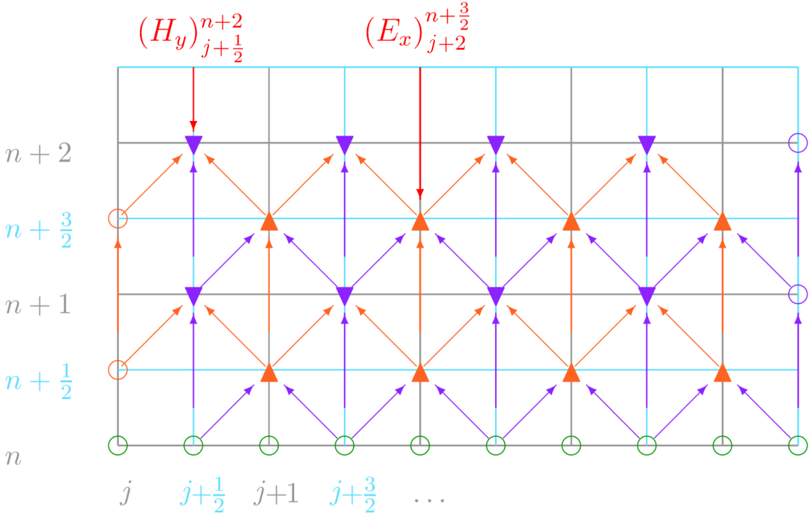
We can discretize these equations according to the Yee Scheme and a central difference approximation for our polynomial chaos system. In our discretization, we choose j to represent the spatial step and n to represent the time step.

$$\frac{H_{j+\frac{1}{2}}^{n+1} - H_{j+\frac{1}{2}}^{n}}{\Delta t} = \frac{1}{\mu_0} \frac{E_{j+1}^{n+\frac{1}{2}} - E_{j}^{n+\frac{1}{2}}}{\Delta z}$$

$$\tilde{\varepsilon}_\infty \frac{E_{j}^{n+\frac{1}{2}} - E_{j}^{n-\frac{1}{2}}}{\Delta t} = -\frac{H_{j+\frac{1}{2}}^{n} - H_{j-\frac{1}{2}}^{n}}{\Delta z} - \sigma \frac{E_{j}^{n+\frac{1}{2}} + E_{j}^{n-\frac{1}{2}}}{2} - \frac{\alpha_{0,j}^{n+\frac{1}{2}} - \alpha_{0,j}^{n-\frac{1}{2}}}{\Delta t}$$

$$\mathrm{A} \frac{\vec{\alpha}_{j}^{n+\frac{1}{2}} - \vec{\alpha}_{j}^{n-\frac{1}{2}}}{\Delta t} + \frac{\vec{\alpha}_{j}^{n+\frac{1}{2}} + \vec{\alpha}_{j}^{n-\frac{1}{2}}}{2} = \frac{\vec{f}_{j}^{n+\frac{1}{2}} + \vec{f}_{j}^{n-\frac{1}{2}}}{2}$$

Yee Scheme in 1-Dimension



Solving for the Update Equations

- Rearranging the first two equations, we obtain the electric and magnetic field updates:

$$E_j^{n+\frac{1}{2}} = \frac{1}{\theta}[\theta^* E_j^{n-\frac{1}{2}} - \frac{2\Delta t}{\Delta z}(H^n_{j+\frac{1}{2}} - H^n_{j-\frac{1}{2}}) - 2(\alpha_{0,j}^{n+\frac{1}{2}} - \alpha_{0,j}^{n-\frac{1}{2}})]$$

$$H^{n+1}_{j+\frac{1}{2}} = -\frac{\Delta t}{\mu_0 \Delta z}[E_{j+1}^{n+\frac{1}{2}} - E_j^{n+\frac{1}{2}}] + H^n_{j+\frac{1}{2}}.$$

where $\theta := 2\tilde{\varepsilon}_\infty + \sigma\Delta t$ and $\theta^* := 2\tilde{\varepsilon}_\infty - \sigma\Delta t$.

- Multiplying both sides of the discretized polynomial chaos system by $2\Delta t$ and rearranging the terms, we see that we can write this system as the matrix equation:

$$(2\mathrm{A} + \Delta t I)\vec{\alpha}_j^{n+\frac{1}{2}} = (2\mathrm{A} - \Delta t I)\vec{\alpha}_j^{n-\frac{1}{2}} + \Delta t I(\vec{f}_j^{n+\frac{1}{2}} + \vec{f}_j^{n-\frac{1}{2}}).$$

Substituting our expression for $E_j^{n+\frac{1}{2}}$ into the first row of our polynomial chaos system and simplifying the algebra we obtain a system of the form

$$(2\tilde{\mathrm{A}} + \Delta t I)\vec{\alpha}_j^{n+\frac{1}{2}} = (2\tilde{\mathrm{A}} - \Delta t I)\vec{\alpha}_j^{n-\frac{1}{2}} + \tilde{f}^n$$

where $\tilde{\mathrm{A}}$ is the matrix A with $\kappa := \frac{\tilde{\varepsilon}_d \Delta t}{\theta}$. added to the (1,1) entry and $\tilde{f}^n$ is the vector,

$$\tilde{f}^n = \begin{pmatrix} (\tilde{\varepsilon}_d \Delta t + \kappa\theta) E_j^{n-\frac{1}{2}} - \frac{2\kappa\Delta t}{\Delta z}(H^n_{j+\frac{1}{2}} - H^n_{j-\frac{1}{2}}) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Sequential Update Equations

- The final form of our sequential update equations are

$$(2\tilde{\mathrm{A}} + \Delta t I)\vec{\alpha}^{n+\frac{1}{2}} = (2\tilde{\mathrm{A}} - \Delta t I)\vec{\alpha}^{n-\frac{1}{2}} + \tilde{f}^{n}$$

$$E_j^{n+\frac{1}{2}} = \frac{1}{\theta}[\theta^* E_j^{n-\frac{1}{2}} - \frac{2\Delta t}{\Delta z}(H^n_{j+\frac{1}{2}} - H^n_{j-\frac{1}{2}}) - 2(\alpha_{0,j}^{n+\frac{1}{2}} - \alpha_{0,j}^{n-\frac{1}{2}})]$$

$$H^{n+1}_{j+\frac{1}{2}} = -\frac{\Delta t}{\mu_0 \Delta z}[E^{n+\frac{1}{2}}_{j+1} - E^{n+\frac{1}{2}}_{j}] + H^n_{j+\frac{1}{2}}.$$

- We use the above equations to program our MATLAB simulation.

The Invertibility of A

Recall that are method is dependent on being able to solve a system of linear equations, expressed in matrix form as:

$$\mathrm{A}\dot{\vec{\alpha}}(t) + \vec{\alpha}(t) = \vec{f}$$

where $\mathrm{A} = r\mathrm{M} + m\mathrm{I}$ and

$$\mathrm{M} = \begin{pmatrix} b_0 & c_1 & 0 & \cdots & 0 \\ a_0 & b_1 & c_2 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & a_{p-2} & b_{p-1} & c_p \\ 0 & \cdots & 0 & a_{p-1} & b_p \end{pmatrix} \quad \text{and} \quad \vec{f} = \begin{pmatrix} \tilde{\varepsilon}_d E \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where a_n, b_n and c_n are the coefficients from the recurrence relation and r and m are scaling constants. Therefore A must be invertable.

The Invertibility of A

We begin by taking the transpose of this matrix, since any matrix is invertible if and only if its transpose is invertible. By the *Levy-Desplanques Theorem* a matrix which is strictly diagonally dominant is non-singular. A matrix is strictly diagonally dominant if

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \text{ for all } i.$$

If $b_i \geq 0$ for all i then $(2\tilde{A} + \Delta t I)^{\mathrm{T}}$ has diagonal entries which are the sum of all non-negative terms. This implies we can ignore all terms containing Δt in order to obtain conditions for arbitrarily small time steps.

The Invertibility of A

This produces the following conditions on A,

$$m > (|a_0| - b_0)\, r$$
$$m > (|c_i| + |a_i| - b_i)\, r \quad \text{for } 0 < i < p$$
$$m > (|c_p| - b_p)\, r.$$

Applying this result to Legendre and Jacobi polynomials yields the condition that $m > r$. The table below gives the bounds for Legendre polynomials.

p	largest root	$m > max(\|c_i\| + \|a_i\| - b_i)r$	$m > (\|c_p\| - b_p)r$
1	$m = 0.577r$	$m > r$	$m > \frac{1}{3}r$
4	$m = 0.906r$	$m > r$	$m > \frac{4}{9}r$
16	$m = 0.991r$	$m > r$	$m > \frac{16}{33}r$
32	$m = 0.997r$	$m > r$	$m > \frac{32}{65}r$

Convergence and Stability

Theorem

Lax-Richtmyer Equivalence Theorem: *A consistent finite difference scheme for a partial differential equation for which the initial value problem is well-posed is convergent if and only if it is stable.*

- A convergent scheme is one that better approximates Maxwell's equations as the spatial and temporal step sizes are decreased.
- Consistency means that the scheme differs from the PDE pointwise by factors that go to zero as Δz and Δt go to zero.
- A finite difference scheme is stable if the errors made at one time step of the calculation do not cause the errors to increase without bound as the computations are continued.

Stability Analysis

Plane wave solutions for our model are of the form:

$$\begin{bmatrix} E_j^n \\ H_j^n \\ \alpha_{0,j}^n \\ \vdots \\ \alpha_{p,j}^n \end{bmatrix} = \begin{bmatrix} \tilde{E} \\ \tilde{H} \\ \tilde{\alpha}_0 \\ \vdots \\ \tilde{\alpha}_p \end{bmatrix} \zeta^n \mathrm{e}^{\mathrm{i}kj\Delta z}$$

Here, $k = \frac{\omega}{c}$ is the wave number, ζ is the complex time eigenvalue, $\vec{x} = [\tilde{E}, \tilde{H}, \tilde{\alpha}_0, \cdots, \tilde{\alpha}_p]^T$ is the corresponding eigenvector, and $\mathrm{e}^{\mathrm{i}kj\Delta z}$ is an eigenfunction of $\frac{\partial}{\partial z}$. In order for our method to be stable we need $|\zeta| \leq 1$.

Stability Analysis

- Performing these substitutions we obtain a system of the form $\mathrm{B}\vec{x} = 0$, where,

$$\mathrm{B} = \left(\begin{array}{ccc|c} \theta\zeta - \theta^* & \frac{4i\rho}{c_\infty}\zeta^{\frac{1}{2}} & 2(\zeta-1) & \\ \frac{2i\rho}{\mu_0 c_\infty}\zeta^{\frac{1}{2}} & \zeta - 1 & 0 & 0 \\ -(\tilde{\varepsilon}_d \Delta t + \kappa\theta^*) & \frac{4i\kappa\rho}{c_\infty}\zeta^{\frac{1}{2}} & 2\kappa(\zeta-1)) & \\ \hline & 0 & & 0 \end{array}\right) + \left(\begin{array}{cc|c} 0 & 0 & \\ 0 & 0 & 0 \\ \hline 0 & & (2\mathrm{A}+\Delta t\mathrm{I})\zeta - (2\mathrm{A}-\Delta t\mathrm{I}) \end{array}\right)$$

  and $\vec{x} = [\tilde{E}, \tilde{H}, \tilde{\alpha}_0, \cdots, \tilde{\alpha}_p]^T$. Additionally, we have used the simplifications $\rho := \nu \sin(\frac{k\Delta z}{2})$ where $\nu := \frac{c_\infty \Delta t}{\Delta z}$.
- This system has a non-trivial solution if and only if $\det \mathrm{B} = 0$. We obtain a characteristic polynomial of the form

$$\sum_{k=0}^{p+3} q_k \zeta^{p-k} = 0$$

Characteristic Polynomial: Legendre Polynomials

If we choose to use Legendre Polynomials with $p = 1$ and neglect conductivity our characteristic polynomial is:

$$q_0\zeta^4 + q_1\zeta^3 + q_2\zeta^2 + q_3\zeta + q_4 = 0,$$

where

$$\begin{aligned}
q_0 &= 4|\mathrm{A}| + 2m\Delta t(\varepsilon_q + 1) + \varepsilon_q \Delta t^2 \\
q_1 &= 16|\mathrm{A}|(\rho^2 - 1) + 4m\Delta t\left[4\rho^2 - \varepsilon_q - 1\right] + 4\rho^2\Delta t^2 \\
q_2 &= 2\Delta t^2(4\rho^2 - \varepsilon_q) - 8|\mathrm{A}|(4\rho^2 - 3) \\
q_3 &= 16|\mathrm{A}|(\rho^2 - 1) - 4m\Delta t\left[4\rho^2 - \varepsilon_q - 1\right] + 4\rho^2\Delta t^2 \\
q_4 &= 4|\mathrm{A}| - 2m\Delta t(\varepsilon_q + 1) + \varepsilon_q \Delta t^2
\end{aligned}$$

Note that $\varepsilon_q = \frac{\varepsilon_s}{\varepsilon_\infty}$, and $|\mathrm{A}| = m^2 - \frac{1}{3}r^2$, where A is the matrix from the polynomial chaos system.

Routh-Hurwitz Stability Criterion

- The Routh-Hurwitz Criterion establishes that the polynomial,

$$f(z) = \sum_{k=0}^{N} b_k z^{N-k}, \qquad b_0 > 0$$

  where b_k are arbitrary constant real coefficients, has no roots in the right-half of the complex-plane only if all the entries of the first column of the Routh Table are nonnegative.
- The transformation

$$\zeta = \frac{z+1}{z-1}$$

  maps the outside of the unit circle to the right half plane. If the transformed polynomial, $f(z)$, has no roots in the right-half plane, then $f(\zeta)$ has no roots outside of the unit circle and our solution is stable.

Constructing the Routh Table

We construct the Routh Table as follows. The coefficients b_k are arranged in two rows, noting that the leading coefficient always appears in the upper left corner:

$$c_{1,k} = b_{2k}, \text{ for } k = 0, ..., \lfloor \frac{N}{2} \rfloor$$
$$c_{2,k} = b_{2k+1}, \text{ for } k = 0, ..., \lfloor \frac{N-1}{2} \rfloor$$

The remaining entries in the the table are obtained using the formula:

$$c_{j,k} = -\frac{1}{c_{j-1,0}} \begin{vmatrix} c_{j-2,0} & c_{j-2,k+1} \\ c_{j-1,0} & c_{j-1,k+1} \end{vmatrix}$$

Transformed Characteristic Polynomial

We consider the case using Legendre Polynomials with p=1. We apply the transformation $\zeta = \frac{z+1}{z-1}$ to the characteristic polynomial and simplify to obtain:

$$\hat{q}_0 z^4 + \hat{q}_1 z^3 + \hat{q}_2 z^2 + \hat{q}_3 z + \hat{q}_4 = 0,$$

where

$$\begin{aligned}
\hat{q}_0 &= \rho^2 \Delta t^2 \\
\hat{q}_1 &= 4\rho^2 m \Delta t \\
\hat{q}_2 &= 4|\mathrm{A}|\rho^2 + (\varepsilon_q - \rho^2)\Delta t^2 \\
\hat{q}_3 &= 2m\Delta t(\varepsilon_q + 1 - 2\rho^2) \\
\hat{q}_4 &= 4|\mathrm{A}|(1-\rho^2)
\end{aligned}$$

Routh Table and Stability Conditions

The Routh Table for our polynomial is:

$\hat{q}_0$	$\hat{q}_2$	$\hat{q}_4$
$\hat{q}_1$	$\hat{q}_3$	0
$q^\star = \frac{\hat{q}_1\hat{q}_2 - \hat{q}_0\hat{q}_3}{\hat{q}_1}$	$\hat{q}_4$	0
$q^{\star\star} = \frac{q^\star\hat{q}_3 - \hat{q}_1\hat{q}_4}{q^\star}$	0	0
$\hat{q}_4$	0	0

By requiring the entries in the first column to be nonnegative we obtain the following stability conditions:

$$m \geq 0$$

$$\nu \leq 1$$

$$\varepsilon_s \geq \varepsilon_\infty$$

Numerical Stability Analysis

We use MAPLE to numerically verify this result by plotting the maximum roots of our characteristic polynomial as a function of $k\Delta z$ for $0 \leq k\Delta z \leq \pi$.

- We first consider the case $p = 1$, $\nu = 1$ for three different temporal resolutions. We define $h := \frac{\Delta t}{\tau}$ so that $\Delta t = h\tau$ and consider the values $h = 0.1$, $h = 0.01$ and $h = 0.001$.
- Next we will consider different spatial resolutions by fixing $h = 0.1$ and varying the value of ν, noting that $\Delta z = \frac{c_\infty h\tau}{\nu}$. We will consider three values of ν where the method is stable and one where the method is unstable.
- Finally we plot the maximum eigenvalues for $\nu = 1$, $h = 0.1$ and various values of p.

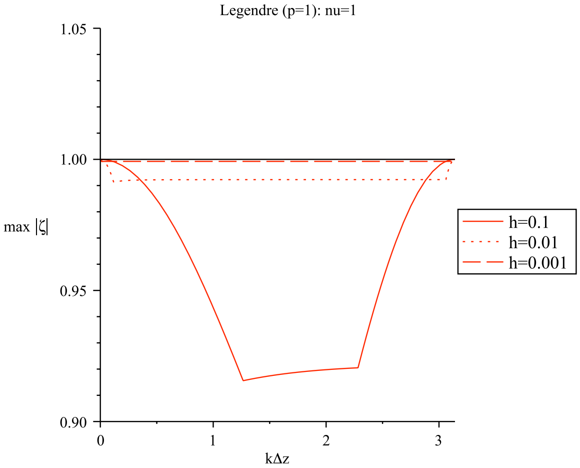


Figure: Maximum eigenvalue versus $k\Delta z$ for $p = 1$, $\nu = 1$ and various temporal resolutions.

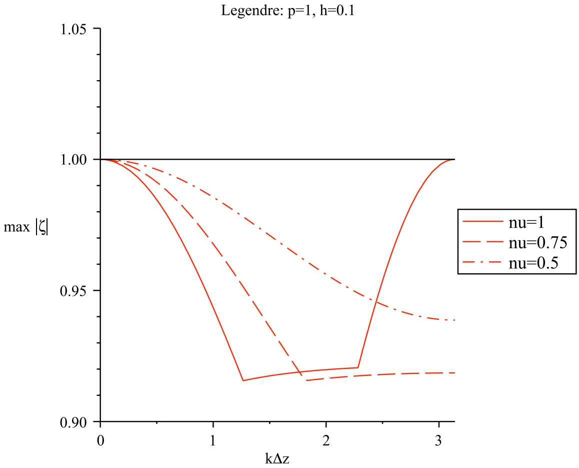


Figure: Maximum eigenvalue versus $k\Delta z$ for $p = 1$, $h = 0.1$ and various spatial resolutions.

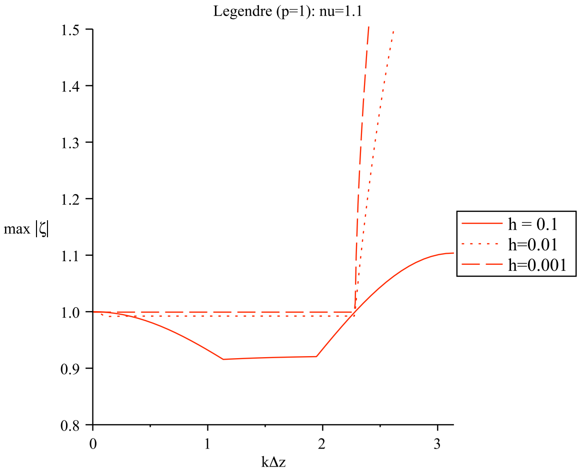


Figure: Maximum eigenvalue versus $k\Delta z$ for $p = 1$, $\nu = 1.1$. This shows that violating the stability condition $\nu \leq 1$ produces eigenvalues outside the unit circle and the method is unstable.

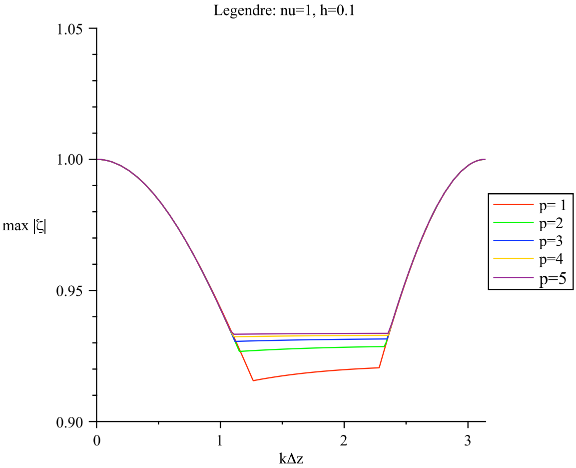


Figure: Maximum eigenvalue versus $k\Delta z$ for $\nu = 1$, $h = 0.1$ and various values of p.

The following values were used for the simulation unless otherwise noted:

$$
\begin{aligned}
r &= \frac{1}{2}\overline{\tau} \\
m &= \overline{\tau} \\
f &= 10\,\mathbf{GHz} \text{ (the frequency of the simulated pulse)} \\
\varepsilon_0 &= 8.85419 \times 10^{-12} \\
\mu_0 &= 4\pi \times 10^{-7} \\
c &= 3 \times 10^{8} \text{ (the speed of light in a vacuum)} \\
\varepsilon_r &= 1 \\
\sigma &= 0 \\
\overline{\tau} &= 8.13 \times 10^{-12} \\
\varepsilon_s &= 80.35 \\
h &= 0.01 \\
\nu &= 0.5 \text{ (the CFL condition)} \\
\Delta t &= h \times \overline{\tau} \text{ (the time step)} \\
\Delta x &= \frac{c\Delta t}{\nu} \text{ (the spacial step)}
\end{aligned}
$$

Most of these parameters have been used in previous years work.

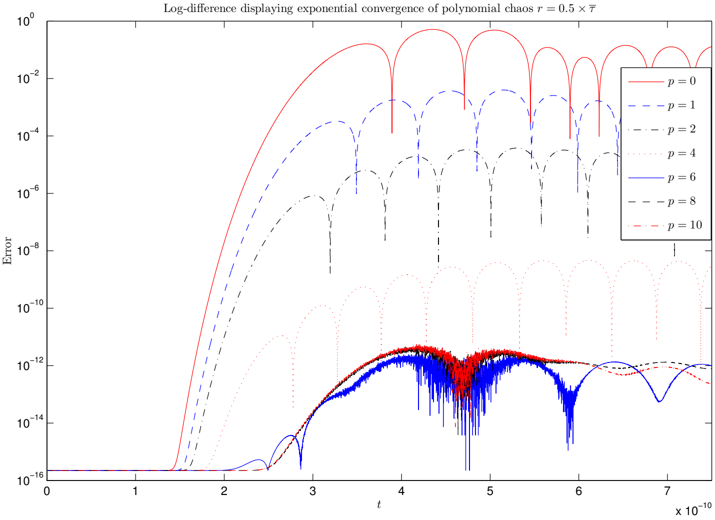


Figure: Error Calculated for various values of p. $r = \frac{1}{2}\overline{\tau}$ and $m = \overline{\tau}$.

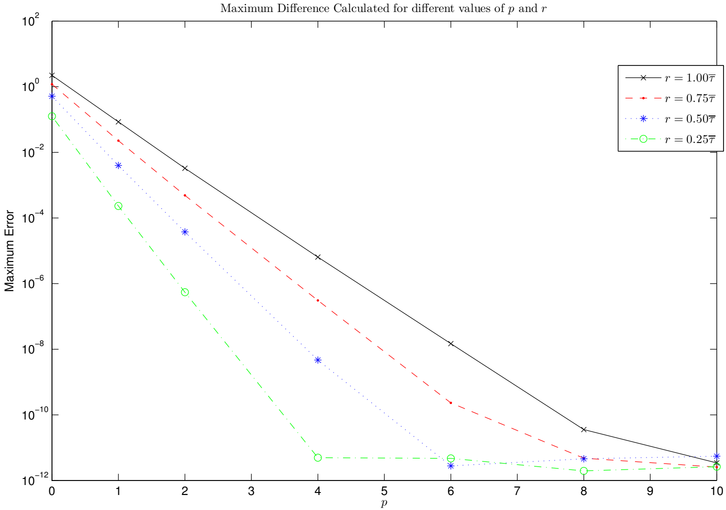


Figure: Maximum Error for various values of p and r, $m = \overline{\tau}$

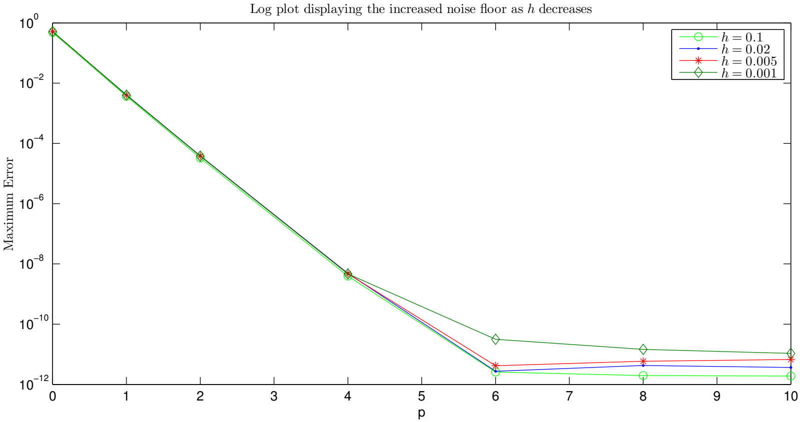


Figure: The increased noise floor for smaller values of Δt

Formulating the Problem

One may use an inverse problem to determine the correct parameters of τ for a two pole Debye model with uniform distributions. Borrowing from their work the procedure is as follows. We begin by defining τ_m to be distributed as,

$$\tau_m \sim \mathscr{U}\left[a_m\overline{\tau}_m, (a_m + b_m)\overline{\tau}_m\right] \tag{1}$$

where a_m and b_m are distribution parameters of τ for which we are concerned. This definition is for a uniform distribution, however, any other distribution could be used in its place. Next we define the stochastic complex permittivity as a function of ω, the angular frequency,

$$\varepsilon(\omega) = \varepsilon_\infty + \sum_m \frac{\varepsilon_{d,m}}{1 + \mathrm{i}\omega\tau_m} \tag{2}$$

The expected value of each pole can be found by integrating with respect to the density function of the distribution being used,

$$\mathbb{E}\left[\varepsilon_m(\omega)\right] = \int_{a_m}^{a_m+b_m} \frac{\varepsilon_{d,m}}{1+\mathrm{i}\omega\tau} f(\tau)\, d\tau \tag{3}$$

For a uniform distribution this becomes,

$$\mathbb{E}\left[\varepsilon_k(\omega)\right] = \frac{1}{b_m}\left[\int_{a_m\overline{\tau}_m}^{(a_m+b_m)\overline{\tau}_m} \frac{\varepsilon_{d,m}}{1-\omega^2\tau^2} d\tau + \mathrm{i}\int_{\tau_{min}}^{\tau_{max}} \frac{-\Delta\varepsilon\omega\tau}{1-\omega^2\tau^2} d\tau\right] \tag{4}$$

$$= \frac{1}{b_m}\frac{\varepsilon_{d,m}}{\omega}\left[arctan(\omega\tau) - \frac{\mathrm{i}}{2} ln\left(1+(\omega\tau)^2\right)\right]_{a_m\overline{\tau}_m}^{(a_m+b_m)\overline{\tau}_m} \tag{5}$$

Summary

- Applying polynomial chaos to the relaxation times of Debye materials results in a more accurate representation.
- This process remains second order accurate in space and time and by the Routh-Hurwitz stability Criterion does not add any more requirements for stability and is gauranteed to have a unique solution.
- This framework displays exponential convergence may be applied to many other distributions and parameters.
- An inverse problem may be formulated to determine the parameters of the distributions for real world simulations.

Future Work

Questions which have remained unanswered

- Under what conditions is the system stable for any value of p?
- Under what conditions is the system stable for other distributions of τ? Most notably the beta distribution.

Future Work

- Formulate an inverse problem to determine the parameters of these distributions for various different materials or to more closely match the Cole-Cole model.
- Update the system to account for multiple poles.