An Analysis of Permanental Ideals Over Hypermatrices

Julia Porcino

April 9, 2011

Permanents

Permanents

- The *permanent* of an $(n \times n)$ square matrix $M = (a_{ij})$ is defined as

  $$\operatorname{perm}(M) = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)},$$

  where S_n is the symmetric group on the set $\{1, \ldots, n\}$.
- Much more computationally difficult then determinants - $O(n2^n)$ vs. $O(n^3)$.

Ideals

Ideals

- R: polynomial ring.
- $I \subseteq R$ is an *ideal* if for all $x \in I$ and $r \in R$, $xr \in I$ and $rx \in I$.

Prime Ideals

- An ideal I is *prime* if it is a proper ideal of R such that for all $x, y \in R$, if $xy \in I$ then either $x \in I$ or $y \in I$.
- Let I be an ideal. A prime ideal P is a *minimal prime ideal* over I if P does not properly contain any other prime ideal that contains I.

Hypermatrices

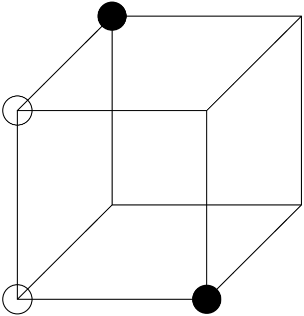

Figure: $M_{2,2,2}$ with two indeterminates distance 1 marked with empty spheres and two indeterminates distance 3 marked with filled-in spheres.

Slice and Diagonal Permanents

Slice Permanent

- $g_{i,a,b} = x_a x_b + x_{s(i,a,b)} x_{s(i,b,a)}$
- Example: $g_{3,(1,1,1),(1,2,2)} = x_{111}x_{122} + x_{112}x_{121}$.

Diagonal Permanent

- $g_{i,a,b} = x_a x_b + x_{s(i,a,b)} x_{s(i,b,a)}$
- Example: $g_{3,(1,1,1),(2,2,2)} = x_{111}x_{222} + x_{112}x_{221}$.

Permanental Ideals

Three kinds of permanental ideals:

- $\hat{J}_2(M_{p,q,r}) =$(diagonal permanents over $M_{p,q,r}$);
- $J_2(M_{p,q,r}) =$(slice permanents over $M_{p,q,r}$);
- $\tilde{J}_2(M_{p,q,r}) =$(diagonal and slice permanents over $M_{p,q,r}$).

Results over $M_{2,2,2}$: $\hat{P}$

Lemma

$\hat{J}_2$ is generated by monomials $x_a x_b$ such that $d(a,b) = 3$.

Theorem

Let $\hat{P}$ be a minimal prime over $\hat{J}_2$. Then $\hat{P}$ is generated by 4 indeterminates, no two of which have distance 3. There are 16 such $\hat{P}$.

Sample $\hat{P}(M_{2,2,2})$

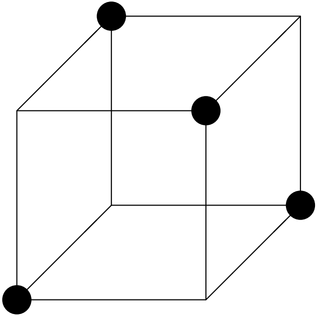

Figure: $M_{2,2,2}$ with the generators of one $\hat{P}$ – $x_{112}, x_{211}, x_{222}, x_{121}$ – marked with filled-in spheres.

Results over $M_{2,2,2}$: P

Theorem

Let P be a minimal prime over J_2. Then P is one of the following:

1. *P is generated by all slice permanents and diagonal determinants;*
2. *P is generated by six indeterminates such that the two indeterminates not in P have distance 3.*

There is one minimal prime of type 1 and 4 minimal primes of type 2.

Results over $M_{2,2,2}$: $\tilde{P}$

Theorem

Let $\tilde{P}$ be a minimal prime over $\tilde{J}_2$. Then $\tilde{P}$ is generated by one slice permanent and all indeterminates outside the permanent. There are six such minimal primes.

Results over $M_{p,q,r}$: $\hat{P}$

Theorem

Let $\hat{P}$ be a minimal prime over $\hat{J}_2$. Then $\hat{P}$ is one of the following:

1. *$\hat{P}$ is generated by all indeterminates from $q-1$ walls;*
2. *$\hat{P}$ is generated by all indeterminates from $r-1$ floors;*
3. *$\hat{P}$ is generated by all indeterminates from $p-1$ faces;*
4. *$\hat{P}$ is generated by all indeterminates except for those indeterminates in exactly one column, one row, and one length such that all three intersect in exactly one place;*
5. *$\hat{P}$ is generated by all indeterminates except four indeterminates such that all four are distance 2 from one another.*

Results over $M_{p,q,r}$: P

Theorem

Define a set S to be admissible if it is a disjoint union of sets of the following form:

- *$s \times t \times u$ such that $s, t, u \in \{1, 2\}$;*
- *$s \times t \times u$ such that two of s, t, u are 1.*

Also, if $a, b \in S$ such that $d(a, b) \leq 2$, a and b are in the same set.

S is maximal admissible if there is no other admissible set that properly contains S.

Let P be a minimal prime over J_2 and S a maximal admissible set. Then P is generated by all indeterminates not in S and all slice permanents and diagonal determinants over S.

Results over $M_{p,q,r}$: $\tilde{P}$

Theorem

Let $\tilde{P}$ be a minimal prime over $\tilde{J}_2$. Then $\tilde{P}$ is one of the following:

1. *$\tilde{P}$ is generated by all indeterminates outside of one column;*
2. *$\tilde{P}$ is generated by all indeterminates outside of one row;*
3. *$\tilde{P}$ is generated by all indeterminates outside of one length;*
4. *$\tilde{P}$ is generated by one slice permanent and all indeterminates outside the permanent.*

Primary Decomposition

A *primary decomposition* of an ideal I is $I = q_1 \cap q_2 \cap \cdots \cap q_s$, where each q_i is a primary ideal.

Preliminary Results:

- Over $M_{2,2,2}$, for all permanental ideals, the primary decomposition does equal the intersection of exactly the minimal primes.

Primary Decomposition

A *primary decomposition* of an ideal I is $I = q_1 \cap q_2 \cap \cdots \cap q_s$, where each q_i is a primary ideal.

Preliminary Results:

- Over $M_{2,2,2}$, for all permanental ideals, the primary decomposition does equal the intersection of exactly the minimal primes.
- Over larger hypermatrices, this also seems to be true. . .

Primary Decomposition

A *primary decomposition* of an ideal I is $I = q_1 \cap q_2 \cap \cdots \cap q_s$, where each q_i is a primary ideal.

Preliminary Results:

- Over $M_{2,2,2}$, for all permanental ideals, the primary decomposition does equal the intersection of exactly the minimal primes.
- Over larger hypermatrices, this also seems to be true…
- Except for J_2.

Generalizing to *n* Dimensions

What constitutes a wall in a fifth-dimension hypercube?

Need a more general way to describe the permanental ideals and their minimal primes.

Let $S \subseteq R$ such that S contains indeterminates from M.

$$Q_S = (x_a \notin S + H(S))$$

$$\hat{Q}_S = (x_a \notin S).$$

Generalizing to n Dimensions

The Next Challenge: Working with More Permanental Ideals

When $n = 4$,

- 4 types of permanents;

Generalizing to n Dimensions

The Next Challenge: Working with More Permanental Ideals

When $n = 4$,

- 4 types of permanents;
- 15 types of permanental ideals.

Thank you!
Questions?