The Math of Magic

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Where It Began

- First appeared in Venice in 1593
  - Orazio Gallasso's *Gicohi di carte bellissimi di regola e di memoria*
- Translated means Beautiful Card Games of Rule and Memory
- Published in *You Can Teach an Old Magician New Tricks* by John P. Bonomo

The 21 Card Trick

- The Magician lays out 21 cards
  - 3 columns or **stacks** of 7 cards, a 3×7 arrangement
  - Every card is located in a specific position
    - Position of key card after k collections denoted p_k
    - In 3×7 arrangement, $0 \leq p_k \leq 20$

Position 0	Position 1	Position 2
Position 3	Position 4	Position 5
Position 6	Position 7	Position 8
Position 9	Position 10	Position 11
Position 12	Position 13	Position 14
Position 15	Position 16	Position 17
Position 18	Position 19	Position 20

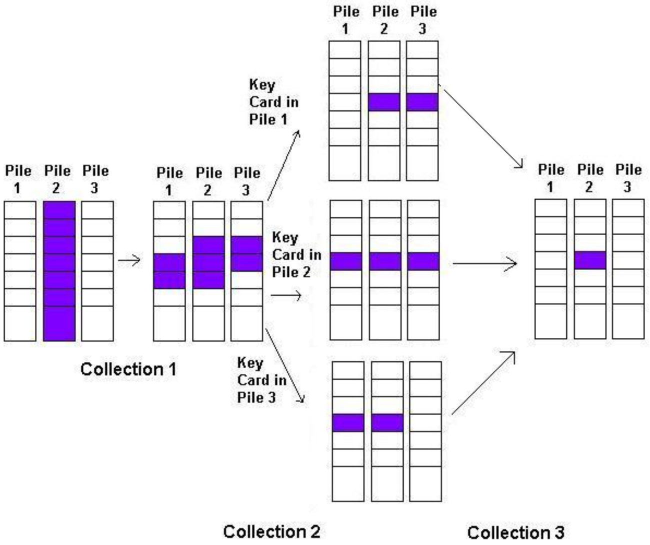
The 21 Card Trick cont.

- Volunteer selects **key card**
- Volunteer signals **key pile**
- Magician collects cards, called **collection**

Collection

- Process repeated twice more
- Magician selects key card

How It Works



Pile 1
Pile 2
Pile 3
Collection 1
Key Card in Pile 1
Key Card in Pile 2
Key Card in Pile 3
Collection 2
Collection 3

Finding the Location Function

- Need to find a function that determines the location of key card after each collection
- We always place 7 cards on top of key pile during collection, this value called **base value**
- Function needed that will add values 0 through 6 to base value, dependent of position of key card

Finding the Location Function

- When $p = 0, 1$, or 2, need to add 0 to base value
- When $p = 3, 4$, or 5, need to add 1 to base value
- etc.

Position 0	Position 1	Position 2
Position 3	Position 4	Position 5
Position 6	Position 7	Position 8
Position 9	Position 10	Position 11
Position 12	Position 13	Position 14
Position 15	Position 16	Position 17
Position 18	Position 19	Position 20

Finding the Location Function - The Floor Function

- Definition: The Floor Function- a function which maps a real number to the preceding largest integer, represented by $\lfloor x \rfloor$. i.e. $\lfloor x \rfloor : x \to \max\{z \in \mathbb{Z} | z \leq x\}$
  - ex. $\lfloor 2.4 \rfloor = 2$
  - ex. $\lfloor \pi \rfloor = 3$
- Consider $\left\lfloor \frac{p}{3} \right\rfloor$, called **offset**
  - $0 \leq p \leq 2, \left\lfloor \frac{p}{3} \right\rfloor = 0$
  - $3 \leq p \leq 5, \left\lfloor \frac{p}{3} \right\rfloor = 1$
  - $6 \leq p \leq 8, \left\lfloor \frac{p}{3} \right\rfloor = 2$
  - etc.

Finding the Location Function - $f(p_0)$

- After first collection, function $f(p) = 7 + \left\lfloor \frac{p}{3} \right\rfloor$
- Use the output of the function $f(p)$ as input for second collection, so position of key card is $f(f(p))$
- Let p_k represent position of key card after k collections
  - $p_2 = f(p_1)$
  - $p_3 = f(p_2)$
- $p_2 = f(p_1) = 7 + \left\lfloor \frac{7+\left\lfloor \frac{p_0}{3} \right\rfloor}{3} \right\rfloor$

Finding the Location Function - Theorem 1

Theorem 1: $\left\lfloor \frac{m+\lfloor x \rfloor}{n} \right\rfloor = \left\lfloor \frac{m+x}{n} \right\rfloor$ **when** m, n **are integers.**

- Examples:
  - $\left\lfloor \frac{7+\lfloor \frac{1}{2} \rfloor}{8} \right\rfloor = \left\lfloor \frac{7+0}{8} \right\rfloor = 0, \left\lfloor \frac{7+\frac{1}{2}}{8} \right\rfloor = 0$
  - $\left\lfloor \frac{3+\lfloor \pi \rfloor}{4} \right\rfloor = \left\lfloor \frac{3+3}{4} \right\rfloor = 1, \left\lfloor \frac{3+\pi}{4} \right\rfloor = 1$

Finding the Location Function - $f(p_1)$

- $p_2 = f(p_1) = 7 + \left\lfloor \frac{7+\lfloor \frac{p_0}{3} \rfloor}{3} \right\rfloor = 7 + \left\lfloor \frac{7+\frac{p_0}{3}}{3} \right\rfloor$ by Theorem 1

- $7 + \left\lfloor \frac{7+\frac{p_0}{3}}{3} \right\rfloor = 7 + 2 + \left\lfloor \frac{1+\frac{p_0}{3}}{3} \right\rfloor = 9 + \left\lfloor \frac{3+p_0}{9} \right\rfloor$

- Our function after the second collection is

  $p_2 = f(p_1) = 9 + \left\lfloor \frac{3+p_0}{9} \right\rfloor$

Finding the Location Function - $f(p_2)$

- $p_3 = f(p_2) = 7 + \left\lfloor \frac{9+\left\lfloor \frac{3+p_0}{9} \right\rfloor}{3} \right\rfloor = 7 + 3 + \left\lfloor \frac{(3+p_0)/(9)}{3} \right\rfloor = 10 + \left\lfloor \frac{3+p_0}{27} \right\rfloor$.

- Function after the third collection is $p_3 = f(p_2) = 10 + \left\lfloor \frac{3+p_0}{27} \right\rfloor$
  - Positions range from 0 to 20, so $0 \le p_0 \le 20$.
  - $\left\lfloor \frac{3+p_0}{27} \right\rfloor$ will never be greater than 0

- Key card will always be placed in position 10

Collection Combinations for 3 Collections

- We can look at all possible placements of the key pile during collections 1, 2, and 3
- Write function dependent on i_k and p_k, where i_k indicates position of key pile during the kth collection
  - $i_k = 0,1$, or 2 if key pile is placed on bottom, middle, or top respectively
- General form of function is $f_{i_k}(p_k) = 7i_k + \left\lfloor \frac{p_k}{3} \right\rfloor$

Collection Combinations for 3 Collections - An Example

- Look at an example where we place the key pile on bottom, top, and middle during first, second, and third collection respectively

- After the first collection, our function is
  $p_1 = f_0(p_0) = 7 \cdot 0 + \left\lfloor \frac{p_0}{3} \right\rfloor = \left\lfloor \frac{p_0}{3} \right\rfloor$
    - We set $i_1 = 0$ because key pile is placed on bottom

- After second collection, our function is
  $p_2 = f_2(p_1) = 7 \cdot 2 + \left\lfloor \frac{\left\lfloor \frac{p_0}{3} \right\rfloor}{3} \right\rfloor = 14 + \left\lfloor \frac{p_0}{9} \right\rfloor.$
    - Set $i_2 = 2$ because key pile is placed on top
    - There will be 14 cards dealt first before key pile is dealt because placed on top

Collection Combinations for 3 Collections - An Example

- After third collection, our function is $p_3 = f_1(p_2) =$
  $7 + \left\lfloor \frac{14+\lfloor \frac{p_0}{9} \rfloor}{3} \right\rfloor = 11 + \left\lfloor \frac{2+\frac{p_0}{9}}{3} \right\rfloor = 11 + \left\lfloor \frac{(18+p_0)/9}{3} \right\rfloor = 11 + \left\lfloor \frac{18+p_0}{27} \right\rfloor$
  - Set $i_3 = 1$ because key pile is placed in middle
  - Because $0 \leq p_0 \leq 20$, if $0 \leq p_0 \leq 8$, $\left\lfloor \frac{18+p_0}{27} \right\rfloor = 0$ yet if $9 \leq p_0 \leq 20$, $\left\lfloor \frac{18+p_0}{27} \right\rfloor = 1$
  - Key card will either be in position 11 or 12 after third collection depending on original position of key card

- Because position of key card after third collection cannot be guaranteed, the function for placement bottom, top, and middle during the first, second, and third collection respectively is **invalid**

Collection Combinations for 3 Collections

- We can look at all collection combinations by choosing placements during the first and second collections and leaving the third collection arbitrary using same methods as above
- Let b, m, t represent placing the key pile on bottom, middle, or top respectively

Collection Combinations for 3 Collections

- The following table represents all possible functions for the various collection combinations

Table 1: Collection Combinations

i_2

		0	1	2
	0	$7i_3 + \left\lfloor \frac{p_0}{27} \right\rfloor$	$7i_3 + 2 + \left\lfloor \frac{9+p_0}{27} \right\rfloor$	$7i_3 + 4 + \left\lfloor \frac{18+p_0}{27} \right\rfloor$
i_1	1	$7i_3 + \left\lfloor \frac{21+p_0}{27} \right\rfloor$	$7i_3 + 3 + \left\lfloor \frac{3+p_0}{27} \right\rfloor$	$7i_3 + 5 + \left\lfloor \frac{12+p_0}{27} \right\rfloor$
	2	$7i_3 + 1 + \left\lfloor \frac{15+p_0}{27} \right\rfloor$	$7i_3 + 3 + \left\lfloor \frac{24+p_0}{27} \right\rfloor$	$7i_3 + 6 + \left\lfloor \frac{6+p_0}{27} \right\rfloor$

Collection Combinations for 3 Collections

- We can the find position of the key card after the third collection for each collection combination

Table 2: Offset Values

i_2

		0	1	2
	0	0	$2,\ 0 \le p_0 \le 17$ $3,\ 18 \le p_0 \le 20$	$4,\ 0 \le p_0 \le 8$ $5,\ 9 \le p_0 \le 20$
i_1	1	$0,\ 0 \le p_0 \le 5$ $1,\ 6 \le p_0 \le 20$	3	$5,\ 0 \le p_0 \le 14$ $6,\ 15 \le p_0 \le 20$
	2	$1,\ 0 \le p_0 \le 11$ $2,\ 12 \le p_0 \le 20$	$3,\ 0 \le p_0 \le 2$ $4,\ 3 \le p_0 \le 20$	6

Probability of Success

- In the previous table, only 3 of the 9 cases were valid
- In the cases that were not valid, the key card will has the possibility of being in two positions, specific for each function
- Can find overall probability of success of choosing the correct key card for all functions if one chooses the most probable position of the key card

$$\frac{3}{9}(1) + \frac{2}{9}\left(\frac{18}{21}\right) + \frac{2}{9}\left(\frac{12}{21}\right) + \frac{2}{9}\left(\frac{15}{21}\right) = \frac{17}{21} \approx 81\%.$$

Adding Collections

- Will adding collections increase probability of successfully choosing the key card?
  - We can look at the 27 different collection combinations, where the fourth collection is arbitrary
  - 81 different collection combinations for 5 collections
  - 243 different collections combinations for 6 collections
    - We will use the methods from earlier to find each of the functions for the different collection combinations

Adding Collections - 4, 5, and 6 Collections

- We can see from the tables that as one adds collections, there continue to be 6 invalid cases
- The probability of success increases with each collection
  - With 4 collections, if one always chooses the most probable location of the key card, the probability of successfully choosing the key card is
    $$\left(\frac{21}{27}\right)+\left(\frac{2}{27}\right)\left(\frac{18}{21}\right)+\left(\frac{2}{27}\right)\left(\frac{12}{21}\right)+\left(\frac{2}{27}\right)\left(\frac{15}{21}\right)=\frac{59}{63},$$
    which is approximately 93.65%.
  - With 5 collections, the probability of success is
    $$\left(\frac{75}{81}\right)+\left(\frac{2}{81}\right)\left(\frac{18}{21}\right)+\left(\frac{2}{81}\right)\left(\frac{12}{21}\right)+\left(\frac{2}{81}\right)\left(\frac{15}{21}\right)=\frac{185}{189},$$
    which is approximately 97.88%.
  - With 6 collections, the probability of success is
    $$\left(\frac{237}{243}\right)+\left(\frac{2}{243}\right)\left(\frac{18}{21}\right)+\left(\frac{2}{243}\right)\left(\frac{12}{21}\right)+\left(\frac{2}{243}\right)\left(\frac{15}{21}\right)=\frac{563}{567},$$
    which is approximately 99.29%.

Theorem 2: In a 3×7 arrangement, given k collections, where $k \geq 3$, then there are a total of 6 invalid collection combinations out of 3^{k-1} total collection combinations

Adding Collections - Theorem 3: The General Case

Theorem 3: In an $n \times m$ card arrangement, with k collections, where $k \geq 3$, and q invalid functions, then there will be q invalid collection combinations of a possible n^{k-1}

Conclusion

- The 21 card trick started in 1593
- One can preform the trick successfully when placing the key pile on bottom bottom, middle middle, and top top during the first and second collections, where the third collection is arbitrary
  - Choosing the most probable position of the key card gives one the overall probability of success approximately 81%
- We can add collections to increase the probability of success
  - As the number of collections increases without bound, the probability of successfully choosing the key card approaches 100%
- Further extensions
  - Is there a way to discover the key card with less collections?
  - One can investigate the infinite case