Tilings with T and skew tetrominoes

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Tileability

Definition (Tiles)

We say that a tile set T **tiles** a region R if R can be completely covered with out overlaps by tiles in T and all tiles used to cover R are contained in the region.

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We say a region R is **tileable** by a tile set T if T tiles R.

Definition (Untileable)

We say a region R is **untileable** by a tile set T if T cannot tile R, i.e. if R is completely covered by tiles in T, then at least two tiles are overlapping or part of at least one tile lays outside the region.

Tiling Problems

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Can a region of some finite area be tiled with a given tile set?

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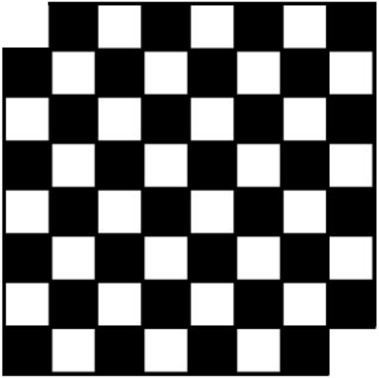
If the region can be tiled, a proof could be as simple as providing a tiling of that region.

If the region cannot be tiled, then the question becomes:
How do you prove a region is untileable.

Using Colors

Can dominoes tiles this modified chessboard?



This question was posed by George Gamov and Marvin Stern in 1958.

Using Modular Arithmetic

	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0
0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0
0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0
0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	

Notice: dominoes will always sum to 1 (mod 2),
the region sums to 0 (mod 2), and
the region has 62 squares.

Definition (Modified Rectangle)

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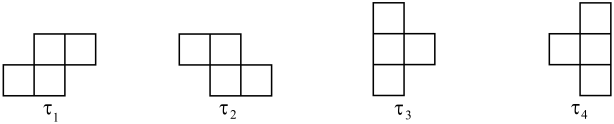

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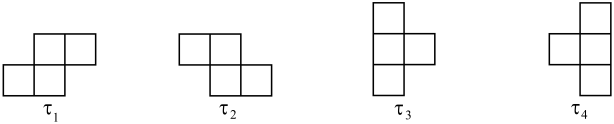

Question: For what values of a and b can L tile M(a,b)?

Theorem

Let a and b be integers strictly greater than 1. Then the set L tiles M(a,b) if and only if either

1. $a \equiv 2$ *(mod 4) and b is odd, or*

2. $b = 2$ *and* $a \equiv 1$ *(mod 4).*

Lemma (1)

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If $a \equiv 2 \pmod{4}$ and $b > 1$ is odd, then L tiles M(a,b).

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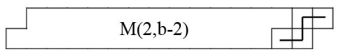

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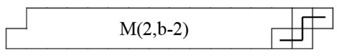

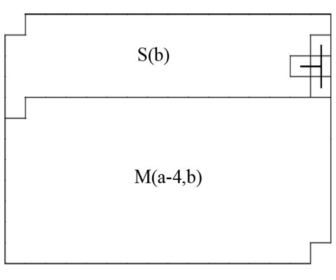

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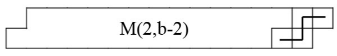

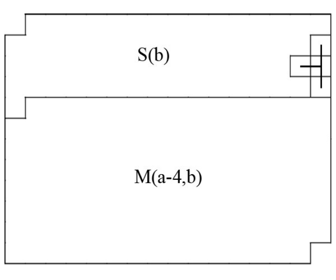

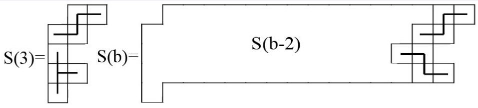

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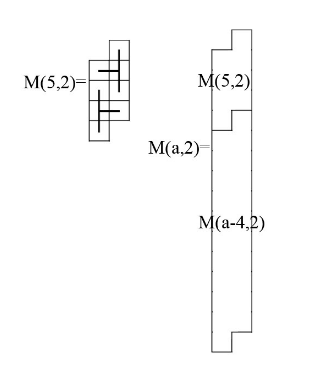

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1. $a \equiv 2 \pmod 4$ and b is odd, or
2. a is odd and $b \equiv 2 \pmod 4$.

We need to show that $b = 2$ and $a \equiv 1 \pmod 4$.

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Proof.

Consider the following coloring:

3	2	3	2	3
1	4	1	4	1
2	3	2	3	2
4	1	4	1	4
3	2	3	2	3

Notice that all the tiles in L sum to zero modulo 5. Thus if L can tile M(a,b) it should sum to zero modulo 5 too, however, it does not. □

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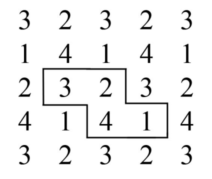

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The Sum of M(a,b) for Lemma (3)

We are going to express the coloring in the following arbitrary terms,

$$\begin{array}{ccccc} C_1 & C_2 & C_1 & C_2 & C_1 \\ C_4 & C_3 & C_4 & C_3 & C_4 \\ C_2 & C_1 & C_2 & C_1 & C_2 \\ C_3 & C_4 & C_3 & C_4 & C_3 \\ C_1 & C_2 & C_1 & C_2 & C_1 \end{array}$$

Where $C_1 + C_2 \equiv 0 \pmod 5$, $C_3 + C_4 \equiv 0 \pmod 5$, and $i, C_i \in \{1, 2, 3, 4\}$.

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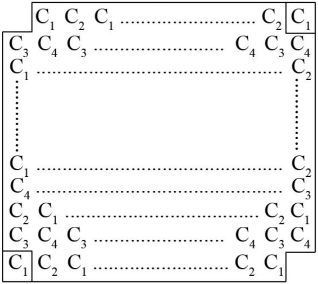
Then the coloring of M(a,b) may be represented as:



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Proof.

Consider the coloring:

2	-1	2	-1	2
-2	1	-2	1	-2
-1	2	-1	2	-1
1	-2	1	-2	1
2	-1	2	-1	2

Notice that all the tiles in L sum to zero. Thus if L can tile M(a,b) it should sum to zero too, however, since $b > 2$ and b is even it does not. □

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Where $C_1 + C_2 + C_3 + C_4 = 0$, $C_1 + C_2 \in \{1, -1\}$, and $C_i \in \{-2, -1, 1, 2\}$ $\forall i$.

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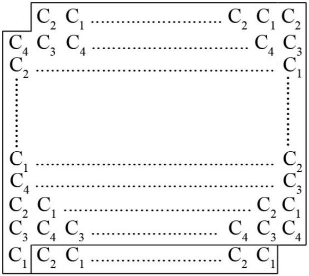

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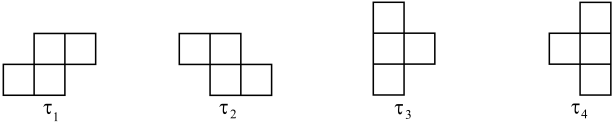

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Thank You

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