Constructing Cayley-Sudoku Tables

Based on "Cosets and Cayley-Sudoku Tables," Mathematics Magazine, April 2010

Kady Hossner
with help from Dr. Michael Ward

April 9, 2011

Outline

What is a Cayley-Sudoku Table?

First, what is a Cayley Table?

Definition

- Named after Arthur Cayley who wrote the first paper on Group Theory
- Essentially an operation table of a group (think multiplication table)

Kady Hossner, with help from Dr. Michael Ward

What is a Cayley-Sudoku Table?

Generically, consider the group $G = \{1, \alpha, \beta, ...\}$

Its Cayley Table would look like this:

	1	α	β	$\ldots$
1	1	α	β	$\ldots$
α	α	α^2	$\alpha\beta$	$\ldots$
β	β	$\beta\alpha$	β^2	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

What is a Cayley-Sudoku Table?

Not generically, lets look at the group $\mathbb{Z}_9 = \{1, 2, 3, 4, 5, 6, 7, 8, 9 = 0\}$ with the operation $+_9$

$9 +_9 2 = 2$ $\quad$ $1 +_9 3 = 4$ $\quad$ $7 +_9 5 = 3$

	9	1	2	3	4	5	6	7	8
9	9	1	(2)	3	4	5	6	7	8
1	1	2	3	(4)	5	6	7	8	9
2	2	3	4	5	6	7	8	9	1
3	3	4	5	6	7	8	9	1	2
4	4	5	6	7	8	9	1	2	3
5	5	6	7	8	9	1	2	3	4
6	6	7	8	9	1	2	3	4	5
7	7	8	9	1	2	(3)	4	5	6
8	8	9	1	2	3	4	5	6	7

What is a Cayley-Sudoku Table?

$9 +_9 2 = 2$ $1 +_9 3 = 4$ $7 +_9 5 = 3$

	9	3	6	1	4	7	2	5	8
9	9	3	6	1	4	7	(2)	5	8
1	1	(4)	7	2	5	8	3	6	9
2	2	5	8	3	6	9	4	7	1
3	3	6	9	4	7	1	5	8	2
4	4	7	1	5	8	2	6	9	3
5	5	8	2	6	9	3	7	1	4
6	6	9	3	7	1	4	8	2	5
7	7	1	4	8	2	5	9	(3)	6
8	8	2	5	9	3	6	1	4	7

What is a Cayley-Sudoku Table?

Formal Definition

A Cayley Sudoku Table is an arrangement of the Cayley table of a group G with composite order $|G| = nk$ that is comprised of smaller $n \times k$ blocks each containing every group element exactly once.

In our example we had $|\mathbb{Z}_9| = 9 = 3 \times 3$, so the blocks in our table were 3×3.
However, the smaller blocks could be 4×6 or 12×35 or 2×208 depending on the group!

Building Cayley-Sudoku Tables

Luckily we don't have to guess at building a table!

Cosets:

Take any subgroup of G, Lets say the subgroup is H and some specific element a of G.

$$\text{Left Coset: } aH := \{ah | \forall h \in H\}$$
$$\text{Right Coset: } Ha := \{ha | \forall h \in H\}$$

Conjugates:

For any $g \in G$, we define H^g (the conjugate) as

$$g^{-1} \star H \star g := \{g^{-1} \star h \star g : h \in H\}$$

Building Cayley-Sudoku Tables

Complete Sets of Left (or Right) Cosets

A C.S.L.C.R. of $H \leq G$ is a set $\{x_1, x_2, ..., x_n\}$ that satisfies

- $\forall x_i = x_j$, $x_i H = x_j H$, and
- $\forall y \in G$, $yH = x_i H$ for some x_i in the set

Example:

$\mathbb{Z}_9$ can be broken up into 3 left cosets

$$<3>:=\{9,3,6\}, \quad 1+<3>:=\{1,4,7\} \text{ and } \quad 2+<3>:=\{2,5,8\}$$

A C.S.L.C.R. could be $\{9,4,2\}$ or $\{3,7,2\}$ *but not* $\{9,4,7\}$

An Easy Way to Build a Table

We are finally ready!:

Construction 1: Let G be a finite group. Assume H is a subgroup of G having order k and the number of distinct cosets is n (so that $|G| = nk$). If $Hg_1, Hg_2, \ldots, Hg_n$ are the n distinct right cosets of H in G, then arranging the Cayley table of G with columns labeled by the cosets $Hg_1, Hg_2, \ldots, Hg_n$ and the rows labeled by sets $T_1, T_2, \ldots, T_k$ (as in the table) yields a Cayley-Sudoku table of G with blocks of dimension $n \times k$ if and only if $T_1, T_2, \ldots, T_k$ partition G into complete sets of left coset representatives of H in G.

	$H \star g_1$	$H \star g_2$	$\ldots$	$H \star g_n$
T_1				
T_2				
$\vdots$				
T_k				

An Easy Way to Build a Table

Example of Construction 1:

- Right Cosets: $<3>:=$ {9,3,6}, $<3>+1:=$ {1,4,7} and $<3>+2;=$ {2,5,8}

An Easy Way to Build a Table

Example of Construction 1:

- Right Cosets: $<3>:=$ {9,3,6}, $<3>+1:=$ {1,4,7} and $<3>+2;=$ {2,5,8}
- C.S.L.C.R.: {9, 1, 2}, {3, 4, 5} and {6, 7, 8}

An Easy Way to Build a Table

Example of Construction 1:

- Right Cosets: $<3>:=$ {9,3,6}, $<3>+1:=$ {1,4,7} and $<3>+2;=$ {2,5,8}
- C.S.L.C.R.: {9, 1, 2}, {3, 4, 5} and {6, 7, 8}
-

	9	**3**	**6**	**1**	**4**	**7**	**2**	**5**	**8**
9	9	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	9
2	2	5	8	3	6	9	4	7	1
3	3	6	9	4	7	1	5	8	2
4	4	7	1	5	8	2	6	9	3
5	5	8	2	6	9	3	7	1	4
6	6	9	3	7	1	4	8	2	5
7	7	1	4	8	2	5	9	3	6
8	8	2	5	9	3	6	1	4	7

An Easy Way to Build a Table

Some things to note about Construction 1:

- $|G|$ must be composite

An Easy Way to Build a Table

Some things to note about Construction 1:

- $|G|$ must be composite
- Any composite group can be made into a Cayley-Sudoku Table

An Easy Way to Build a Table

Some things to note about Construction 1:

- $|G|$ must be composite
- Any composite group can be made into a Cayley-Sudoku Table
- In our example Right and Left Cosets were the same (i.e. the subgroup was normal), but they don't have to be

A Not-So-Easy Way to Build a Table

Construction 2: Assume H is a subgroup of G having order k and index n. Also suppose $t_1H, t_2H, \ldots, t_nH$ are the distinct left cosets of H in G. Arranging the Cayley table of G with columns labeled by the cosets $t_1H, t_2H, \ldots, t_nH$ and the rows by sets $L_1, L_2, \ldots, L_k$ yields a Cayley Sudoku table of G with blocks of dimension $n \times k$ if and only if $L_1, L_2, \ldots, L_k$ are complete sets of left coset representatives of H^g for all $g \in G$.

	t_1H	t_2H	$\ldots$	t_nH
L_1				
L_2				
$\vdots$				
L_k				

A Not-So-Easy Way to Build a Table

An Example

Consider the Group S_3 and the Subgroup $H = \{\varepsilon, (1\,2)\}$

Left Cosets of H:

- $(12)H = \{(12)\varepsilon, (12)(12)\} = \{\varepsilon, (12)\} = \varepsilon H$
- $(13)H = \{(13)\varepsilon, (13)(12))\} = \{(13), \varepsilon\} = (123)H$
- $(23)H = \{(23)\varepsilon, (23)12)\} = \{(23), (132)\} = (132)H$

A Not-So-Easy Way to Build a Table

An Example

Conjugates

- $H^{\varepsilon} = \varepsilon^{-1} H \varepsilon = \{\varepsilon, (12)\}$
- $H^{(123)} = (123)^{-1} H (123) = \{(132)\varepsilon(123), (132)(12)(123)\} = \{\varepsilon, (13)\}$
- $H^{(13)} = \{\varepsilon, (32)\}$
- $H^{(12)} = \{\varepsilon, (12)\} = H$
- $H^{(23)} = \{\varepsilon, (13)\} = H^{(123)}$
- $H^{(132)} = \{\varepsilon, (32)\} = H^{(13)}$

A Not-So-Easy Way to Build a Table

An Example

A Not-So-Easy Way to Build a Table

An Example

- Left Cosets of H^{ε}

A Not-So-Easy Way to Build a Table

An Example

- Left Cosets of H^{ε}
  - $(12)H = \{\varepsilon, (12)\}$
  - $(13)H = \{(123), (13)\}$
  - $(23)H = \{(23), (132)\}$

A Not-So-Easy Way to Build a Table

An Example

- Left Cosets of H^{ε}
  - $(12)H = \{\varepsilon, (12)\}$
  - $(13)H = \{(123), (13)\}$
  - $(23)H = \{(23), (132)\}$
- Left Cosets of $H^{(123)}$

A Not-So-Easy Way to Build a Table

An Example

- Left Cosets of H^{ε}
  - $(12)H = \{\varepsilon, (12)\}$
  - $(13)H = \{(123), (13)\}$
  - $(23)H = \{(23), (132)\}$
- Left Cosets of $H^{(123)}$
  - $(13)H^{(123)} = \{\varepsilon, (13)\}$
  - $(23)H^{(123)} = \{(23), (123)\}$
  - $(12)H^{(123)} = \{(12), (132)\}$

A Not-So-Easy Way to Build a Table

An Example

- Left Cosets of H^{ε}
  - $(12)H = \{\varepsilon, (12)\}$
  - $(13)H = \{(123), (13)\}$
  - $(23)H = \{(23), (132)\}$
- Left Cosets of $H^{(123)}$
  - $(13)H^{(123)} = \{\varepsilon, (13)\}$
  - $(23)H^{(123)} = \{(23), (123)\}$
  - $(12)H^{(123)} = \{(12), (132)\}$
- Left Cosets of $H^{(13)}$

A Not-So-Easy Way to Build a Table

An Example

- Left Cosets of H^{ε}
  - $(12)H = \{\varepsilon, (12)\}$
  - $(13)H = \{(123), (13)\}$
  - $(23)H = \{(23), (132)\}$
- Left Cosets of $H^{(123)}$
  - $(13)H^{(123)} = \{\varepsilon, (13)\}$
  - $(23)H^{(123)} = \{(23), (123)\}$
  - $(12)H^{(123)} = \{(12), (132)\}$
- Left Cosets of $H^{(13)}$
  - $(23)H^{(13)} = \{\varepsilon, (23)\}$
  - $(12)H^{(13)} = \{(12), (123)\}$
  - $(13)H^{(13)} = \{(13), (132)\}$

A Not-So-Easy Way to Build a Table

An Example

- Left Cosets of H^{ε}
  - $(12)H = \{\varepsilon, (12)\}$
  - $(13)H = \{(123), (13)\}$
  - $(23)H = \{(23), (132)\}$
- Left Cosets of $H^{(123)}$
  - $(13)H^{(123)} = \{\varepsilon, (13)\}$
  - $(23)H^{(123)} = \{(23), (123)\}$
  - $(12)H^{(123)} = \{(12), (132)\}$
- Left Cosets of $H^{(13)}$
  - $(23)H^{(13)} = \{\varepsilon, (23)\}$
  - $(12)H^{(13)} = \{(12), (123)\}$
  - $(13)H^{(13)} = \{(13), (132)\}$
- According to our definition, our row blocks will be labeled by a C.S.L.C.R.

A Not-So-Easy Way to Build a Table

An Example

- Left Cosets of H^{ε}
  - $(12)H = \{\varepsilon, (12)\}$
  - $(13)H = \{(123), (13)\}$
  - $(23)H = \{(23), (132)\}$
- Left Cosets of $H^{(123)}$
  - $(13)H^{(123)} = \{\varepsilon, (13)\}$
  - $(23)H^{(123)} = \{(23), (123)\}$
  - $(12)H^{(123)} = \{(12), (132)\}$
- Left Cosets of $H^{(13)}$
  - $(23)H^{(13)} = \{\varepsilon, (23)\}$
  - $(12)H^{(13)} = \{(12), (123)\}$
  - $(13)H^{(13)} = \{(13), (132)\}$
- According to our definition, our row blocks will be labeled by a C.S.L.C.R.
- Our first block will be $\varepsilon, (123), (132)$

A Not-So-Easy Way to Build a Table

An Example

- Left Cosets of H^{ε}
  - $(12)H = \{\varepsilon, (12)\}$
  - $(13)H = \{(123), (13)\}$
  - $(23)H = \{(23), (132)\}$
- Left Cosets of $H^{(123)}$
  - $(13)H^{(123)} = \{\varepsilon, (13)\}$
  - $(23)H^{(123)} = \{(23), (123)\}$
  - $(12)H^{(123)} = \{(12), (132)\}$
- Left Cosets of $H^{(13)}$
  - $(23)H^{(13)} = \{\varepsilon, (23)\}$
  - $(12)H^{(13)} = \{(12), (123)\}$
  - $(13)H^{(13)} = \{(13), (132)\}$
- According to our definition, our row blocks will be labeled by a C.S.L.C.R.
- Our first block will be $\varepsilon, (123), (132)$
- Our second block will be $(12), (13), (23)$

A Not-So-Easy Way to Build a Table

An Example

	ε	(12)	(123)	(13)	(23)	(132)
ε	ε	(12)	(123)	(13)	(23)	(132)
(123)	(123)	(13)	(132)	(23)	(12)	ε
(132)	(132)	(23)	ε	(12)	(13)	(123)
(12)	(12)	ε	(23)	(132)	(123)	(13)
(13)	(13)	(123)	(12)	ε	(132)	(23)
(23)	(23)	(132)	(13)	(123)	ε	(12)

A Not-So-Easy Way to Build a Table

Some things to note about Construction 2:

A Not-So-Easy Way to Build a Table

Some things to note about Construction 2:

- $|G|$ must be composite

A Not-So-Easy Way to Build a Table

Some things to note about Construction 2:

- $|G|$ must be composite
- If the supgroup is normal, then Construction 2 is the same as Construction 1

A Not-So-Easy Way to Build a Table

Some things to note about Construction 2:

- $|G|$ must be composite
- If the supgroup is normal, then Construction 2 is the same as Construction 1
- ***Does not work with every group***

A Not-So-Easy Way to Build a Table

Conditions for Construction 2

- When H is a normal subgroup of G, $H^{\epsilon} = H$

A Not-So-Easy Way to Build a Table

Conditions for Construction 2

- When H is a normal subgroup of G, $H^{\varepsilon} = H$
- When $G := \{th : t \in T, h \in H\} := TH$ and $T \cap H = \emptyset$

A Not-So-Easy Way to Build a Table

Conditions for Construction 2

- When H is a normal subgroup of G, $H^{\varepsilon} = H$
- When $G := \{th : t \in T, h \in H\} := TH$ and $T \cap H = \emptyset$
- When G has exactly 2 conjugates

A Not-So-Easy Way to Build a Table

Conditions for Construction 2

- When H is a normal subgroup of G, $H^{\epsilon} = H$
- When $G := \{th : t \in T, h \in H\} := TH$ and $T \cap H = \emptyset$
- When G has exactly 2 conjugates
- Maybe more?

Using Loops to Build Cayley-Sudoku Tables

Definition

A Loop is a set L with an operation $\star$ s.t.

- $\star$ has an identity in L
- $\forall a, b \in L$ the equations:
  $a \star x = b$ and
  $y \star a = b$
  have unique Solutions

Every group is a Loop, but not vice-versa

Using Loops to Build Cayley-Sudoku Tables

Assume L with $\star$ is a Loop.

Fix $x \in L$ and define $\lambda_x : L \to L$ as:

$$\lambda_x(y) = x \star y, \forall y \in L$$

$\Lambda = \{\lambda_x \ : \ x \in L\}$

$G =< \Lambda >$ (smallest subgroup of $S_{|L|}$ containing Λ)

$G_e := \{\alpha \in G \ : \ \alpha(e) = \varepsilon\}$ called "stabilizer of e"

Using Loops to Build Cayley-Sudoku Tables

Taking G and G_e we satisfy the conditions of a 1939 R. Baer's Theory for Loops:

R. Baer Loop Theory

Let G be a group, H a subgroup of G, a C.S.L.C.R. $\{x_1, x_2, ..., x_k\}$ of H in G satisfies the condition $\underline{\forall xH, yH \in G/H \text{ and } \exists! x_i}$ $\underline{\text{s.t. } x_i xH = yH}$ if and only if $\{x_1, x_2, ..., x_k\}$ is a C.S.L.C.R. of H^g $\forall g \in G$.

Using Loops to Build Cayley-Sudoku Tables

So does this give a new condition where Construction 2 works?

Maybe

It has the potential to

Concluding Remarks

- We can see from R. Baer's Theorem that building Cayley-Sudoku tables has its roots in Loop Theory.

Concluding Remarks

- We can see from R. Baer's Theorem that building Cayley-Sudoku tables has its roots in Loop Theory.
- When solving Sudoku Tables, it can help to look for a group identity

Concluding Remarks

- We can see from R. Baer's Theorem that building Cayley-Sudoku tables has its roots in Loop Theory.
- When solving Sudoku Tables, it can help to look for a group identity
- Most Sudoku Tables are not $\mathbb{Z}_9$ tables

Concluding Remarks

Questions?

Concluding Remarks

References

1. Carmichael, Jennifer, Keith Schloman and Michael B. Ward. *Cosets and Cayley – Sudoku Tables*. Mathematics Magazine vol. 83, 130-139, 2010.
2. Denes, J. and A.D. Keedwell. *Latin Squares and their Applications*. Academic Press Inc, New York, 1974.
3. Denes, J. *Algrbraic and Combinatoral Characterzations of Latin Squares*. Mathematica Slovaca, Vol. 17, No. 4, 249-265, 1967.
4. Mirsky, Leonid. *Transversal Theory* Academic Press Inc, New York, 1971.
5. Nagy, Peter T. and Karl Strambach. *Loops in Group Theory and Lie Theory*. Walter De Gruyter and Co. Berlin, 2002.
6. Wall, D.W. *Sub – Quasigroups of Finite Quasigroups*. Pacific Journal of Mathematics Vol. 7, 1711-1714, 1957.