

Solow Growth Model: Exposition

- Model grew out of work by Robert Solow (and, independently, Trevor Swan) in 1956.
- Describes how “natural output” (Y_n assuming full efficiency) evolves in an economy with a constant saving rate
- Key question: Can an economy sustain perpetual growth in per-capita income through ongoing increases in capital? (Answer: No)

Aggregate production function

- The center-piece of every growth model is the aggregate production function
- Does an aggregate production function exist?
 - Yes, if all firms have constant returns to scale and face the same prices for labor and capital.
- In Solow model, we write as $Y(t) = F[K(t), A(t)L(t)]$
 - We use (t) notation because we are working in continuous time
 - See Coursebook Chapter 2 for details
 - We will suppress the time dependence when it isn't needed
 - $A(t)$ is an index of technology or productivity
 - We model as “Harrod neutral” because it is convenient and leads to reasonable conclusions
- Conditions on production function
 - MPK is positive and diminishing
 - $MPK = F_K(K, AL) > 0$
 - Briefly discuss partial derivative and what it means
 - $F_{KK}(K, AL) < 0$
 - MPL is positive and diminishing
 - $MPL = F_L(K, AL) > 0$
 - $F_{LL}(K, AL) < 0$
 - Increase in K raises MPL (and vice versa): $F_{KL}(K, AL) > 0$
 - Constant returns to scale:
 - $F(cK, cAL) = cF(K, AL), \quad \forall c > 0$
- Intensive form of production function
 - Since c can be any positive number, let $c = 1/AL$
 - $F\left(\frac{K}{AL}, 1\right) = \frac{1}{AL} F(K, AL) = \frac{Y}{AL}$
 - AL is the amount of “effective labor” or the amount of labor measured in efficiency units

- This is not important for itself, but is a useful analytical magnitude.
- For interpretation purposes, we will be more concerned with the behavior of Y and Y/L than with Y/AL .
- Define $y \equiv \frac{Y}{AL}$ and $k \equiv \frac{K}{AL}$, and let $f(\bullet) = F(\bullet, 1)$
 - Then $y = f(k)$ expresses the “intensive” form of production function
 - $MPK = f'(k) > 0$
 - $f''(k) < 0$
 - Graph of intensive production function is increasing at a decreasing rate
- Inada conditions
 - $\lim_{k \rightarrow 0} f'(k) = \infty$ assures that intensive production function is vertical as it leaves the origin: MPK is infinitely large if we have no capital and finite labor.
 - $\lim_{k \rightarrow \infty} f'(k) = 0$ assures that the intensive production function eventually becomes horizontal as k increases to infinity: MPK becomes zero as capital is super-abundant.

Equations of motion and structure of economy

- Labor supply grows at constant exogenous (continuously compounded) rate n
 - $\dot{L}(t) = nL(t), \quad \frac{\dot{L}(t)}{L(t)} = n.$
 - $L(t) = L(0)e^{nt}$
- Technology/productivity improves at constant exogenous rate g
 - $\dot{A}(t) = gA(t), \quad \frac{\dot{A}(t)}{A(t)} = g.$
 - $A(t) = A(0)e^{gt}$
- Output is used for consumption goods and investment in new capital (no government spending, closed economy)
 - $Y(t) = C(t) + I(t)$
- Households allocate their income between consumption and saving
 - $Y(t) = C(t) + S(t), \quad I(t) = S(t)$
- Capital accumulates over time through investment and depreciates at a constant proportional rate δ
 - $\dot{K}(t) = I(t) - \delta K(t)$

- Key assumption: Saving is constant share of income s :
 - $S(t) = sY(t)$, so $\dot{K}(t) = sY(t) - \delta K(t)$

Solow's central question

- Using the equations of the model: $Y \uparrow \Rightarrow S \uparrow \Rightarrow I \uparrow \Rightarrow K \uparrow \Rightarrow Y \uparrow$
 - Can this process lead to sustained growth in output forever?
 - Can “capital deepening” alone lead to eternal improvements in living standards?
- To anticipate the result of our analysis: No.
 - Given the path of labor input, increases in capital lead to decreasing effects on output because we have assumed diminishing marginal returns to capital
- If we had a plausible model in which marginal returns to “capital” were not diminishing, then the answer could reverse.
 - Centuries of economic analysis uses “law” of diminishing marginal returns and evidence seems supportive.
 - Is it plausible for marginal returns to be non-diminishing?
 - Perhaps for an augmented concept of capital
 - Modern theories of “endogenous growth” consider human capital and knowledge capital along with physical capital
 - These theories (discussed in Romer's Chapter 3) allow for non-diminishing returns to a broadened concept of capital and change the answer to Solow's question

Solow Growth Model: Steady-State Growth Path

Concepts of dynamic equilibrium

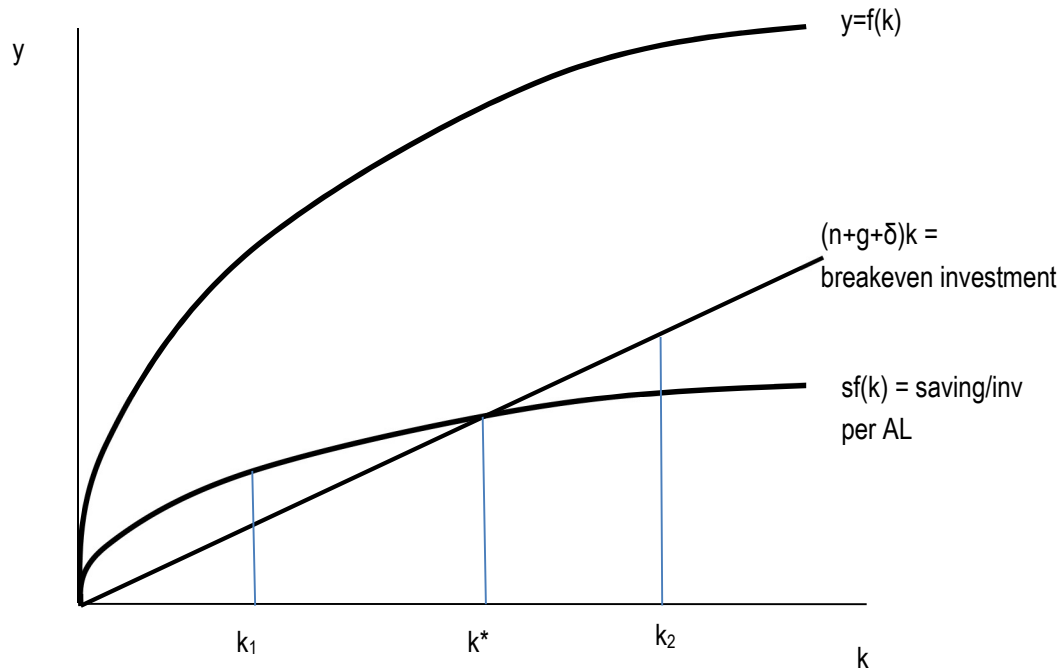
- What is an appropriate concept of equilibrium in a model where variables like Y and K grow over time?
 - Must consider a growth path rather than a single, constant equilibrium value
 - Stable equilibrium growth path is one where
 - If the economy is on the equilibrium path it will stay there
 - If the economy is off the equilibrium path it will return to it
- Equilibrium growth path could be constant K , constant rate of growth of K , or something completely different (oscillations, explosive/accelerating growth, decay to zero, etc.)
 - We build on the work of Solow and others who determined the nature of the equilibrium growth path for our models.
 - As long as we can demonstrate existence and stability, we know we have solved the problem.
- In Solow model (and others), the equilibrium growth path is a *steady state* in which “level variables” such as K and Y grow at constant rates and the ratios among key variables are stable.
 - I usually call this a “steady-state growth path.”
 - Romer tends to use “balanced growth path” for the same concept.

Finding the Solow steady state

- In the Solow model, we know that L grows at rate n and A grows at rate g . The growth of K is determined by saving. Since Y depends on K , AL , it seems highly unlikely that output is going to be unchanging in steady state (a “stationary state”).
- Easiest way to characterize Solow steady state is as a situation where y and k are constant over time.

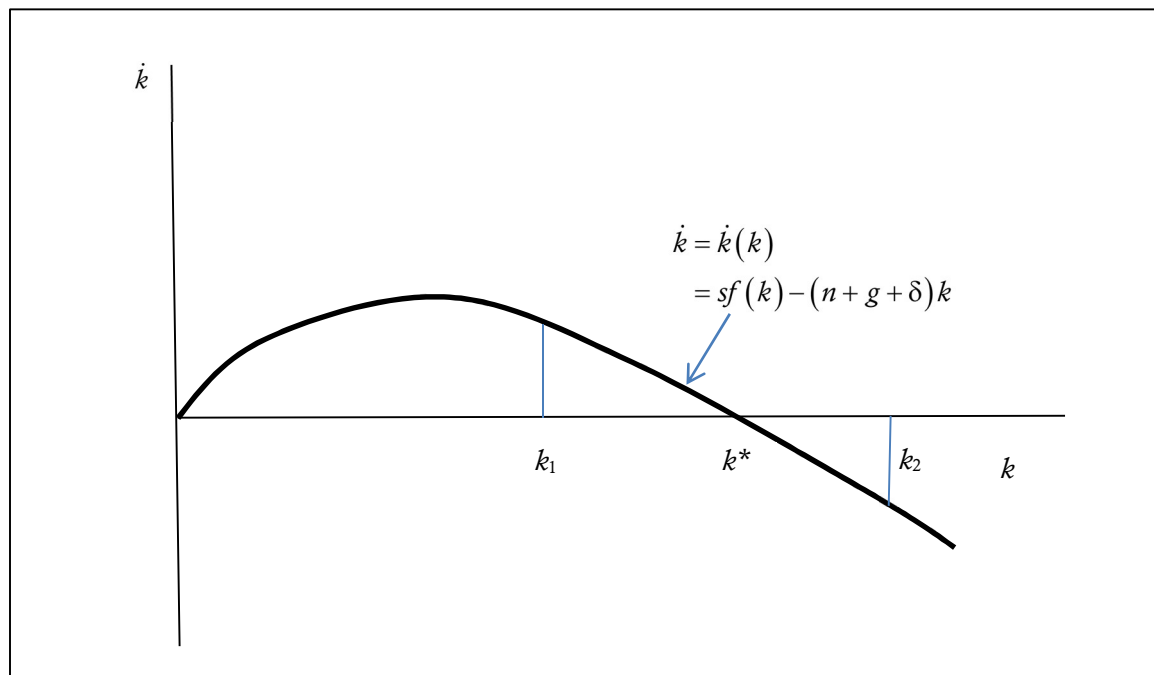
- Since $k = \frac{K}{AL}$, $\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - g - n$, so if k is unchanging, $\dot{k} = 0$ and K must be growing at rate $g + n$.
- Using the equation above and substituting for \dot{K} yields
 - $$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - g - n = \frac{sY - \delta K}{K} - g - n$$
 - $$= \frac{sY}{K} - \delta - g - n = \frac{sy}{k} - \delta - g - n = \frac{sf(k)}{k} - \delta - g - n.$$
 - $$\dot{k} = sf(k) - (g + n + \delta)k$$
 - This is the central equation of motion for the Solow model

- Graph in terms of y and k :



- Breakeven investment line:
 - How big a flow of new capital per unit of effective labor is necessary to keep existing K/AL constant?
 - Must offset shrinkage in numerator through depreciation and increase in denominator through labor growth and technological progress:
 - Need δ for each unit of k to replace depreciating capital
 - Need n for each unit of k to equip new workings
 - Need g for each unit of k to “equip” new technology
 - The more capital each effective labor unit has the bigger the new flow of capital that is required to sustain it: breakeven investment is linear in capital per effective worker.
- At k_1 the amount of new investment per effective worker (on curve) exceeds the amount required for breakeven (on the line) by the gap between the curve and the line, so k is increasing ($\dot{k} > 0$).
- At k_2 the amount of new investment per effective worker falls short of the amount required for breakeven, so k is decreasing ($\dot{k} < 0$).
- At k^* the amount of new investment per effect worker exactly balances the need for breakeven investment, so k is stable: $\dot{k} = 0$.
 - At this level of k the economy has settled into a steady state in which k will not change.

- Show graph with \dot{k} on vertical axis.



- In this graph, k_1 and k_2 have same interpretation as in earlier graph.
- Existence and stability
 - Will there always be a single, unique intersection of the line and curve?
 - Yes.
 - Diminishing returns assumption assures that curve is concave downward.
 - Inada conditions assure that curve is vertical at origin and horizontal in limit.
 - For any finite slope of the breakeven line, there will be one intersection with curve.
 - Because $k < k^* \Rightarrow \dot{k} > 0$ and $k > k^* \Rightarrow \dot{k} < 0$, if economy begins at any level of k other than k^* it will converge over time toward k^* .
 - Steady-state growth path exists, is unique, and is stable.

Characteristics of steady-state growth path

- We now consider the behavior of macroeconomic variables when a Solow economy is on its steady-state growth path. These are the crucial outcomes of the Solow analysis.
- k and y are constant
 - Capital per effective labor unit k is unchanging over time in the steady state.
 - Since output per effective labor unit y depends on k through the production function, it is also unchanging.
- K grows at rate $n + g$

- $\frac{\dot{A}}{A} = g$, $\frac{\dot{L}}{L} = n$, so AL grows at $n + g$.
- $K \equiv kAL$ must grow at $n + g$ (or numerator and denominator of k must grow at same rate for it to stay constant)
- Y grows at rate $n + g$
 - Can make the same argument for y and Y as for k and K
 - Alternatively, $Y = F(K, AL)$ with constant returns to scale. Both K and AL are growing at $n + g$, so each factor expands by $n + g$ each year and total output must also expand at $n + g$.
- Y/L grows at g
 - This is the most important of the outcomes for standards of living.
 - Y grows at $n + g$ and L grows at n , so the quotient grows at the difference: g
- This means that in the steady state, **living standards (output per person) grow at the rate of technological progress.**
 - Capital deepening alone cannot sustain ongoing growth in per-capita output
 - Only technological progress can lead to improvements in output per worker
 - Growth in this model is “exogenous”: output growth = $n + g$, both of which are taken as given from outside the model

Determinants of steady-state path

What determines the value of k^* and therefore y^* ? Anything that shifts the curve or the line in the diagram will change k^* and y^* .

- **Increase in s**
 - Intuitively: we think that an increase in saving should lead to more capital accumulation and higher steady-state k and y
 - Graphically: An increase in s shifts $sf(k)$ curve upward, leading to $\dot{k} > 0$ at original k^* and movement to the right
 - Economy converges to a new steady-state growth path with a higher k^* and y^*
 - However, note that the growth rate on the new path is still the same: K and Y grow at $n + g$ and Y/L grows at g
 - Changes in the saving rate have a “level effect” not a “growth effect” in the Solow model
 - Show parallel growth paths
- **Increase in δ**
 - Intuitively: we think that more depreciation should lead to less capital accumulation and lower steady-state k and y
 - Graphically: An increase in δ makes the break-even line $(n + g + \delta)k$ steeper, leading to $\dot{k} < 0$ at original k^* and movement to the left

- Economy converges to new steady-state growth path with a lower k^* and y^*
- Again, these are level effects, not growth effects: the growth rates are the same
- **Increase in n**
 - Intuitively: More rapid population growth should allow economy to grow faster because labor input is growing faster, but given the saving rate it will be harder to accumulate capital per worker because the higher birth rate means more new workers must be equipped
 - Graphically: Higher n makes the break-even line steeper, leading to $\dot{k} < 0$ at original k^* and movement to the left
 - Economy converges to a new steady-state growth path with a lower k^* and y^*
 - However, the growth rate of Y and K on this new path is greater than the original growth rate because $n + g$ has increased.
 - Show new path for Y that is lower but steeper than old one.
 - Growth rate of Y/L has not changed: it still grows at g .
 - Level effect causes new growth path for Y/L to be parallel but lower
- **Increase in g**
 - Intuitively: Faster technological progress should allow economy to grow faster, both in aggregate and in per-capita terms.
 - Less intuitively: Because of the way that the production function incorporates technology, an increase in technological progress means that more investment is needed to keep up with the growth in AL , thus making it harder to accumulate capital vis a vis AL .
 - Graphically: Higher g makes the break-even line steeper, leading to $\dot{k} < 0$ at original k^* and movement to the left
 - Economy converges to a new steady-state growth path with a lower k^* and y^*
 - Just as with increase in n , the growth rate of Y and K on this new path is greater than the original growth rate because $n + g$ has increased.
 - Show new path for Y that is lower but steeper than old one.
 - Now the growth rate of Y/L has increased as well.
 - Its new path is also steeper but starting from a lower level.

Golden-Rule growth path (Save for problem set)

Given that the saving rate affects k^* and y^* , we might consider asking the question “What is the optimal saving rate?”

- What do we mean by “optimal”?
 - We don’t have utility functions, so we cannot really conduct welfare analysis.
 - Is highest possible y^* optimal?
 - This would imply $s = 1$ is best.

- If $s = 1$, then consumption = 0, which doesn't seem like high utility
 - Perhaps it makes sense to maximize C/L ?
 - Yes, because consumption yields utility
 - But what C/L ?
 - Setting $s = 0$ maximizes current consumption given the current capital stock, but k will fall over time so y will fall and C cannot be maintained
 - Golden-Rule criterion: ignore current consumption and focus on sustainable steady-state level: maximize path of $(C/L)^*$
 - This is called the “Golden Rule” path because it gives equal priority to future generations
 - Given that A is exogenous, we can maximize $c^* = (C/AL)^*$ and it leads to the same result.
- Graphical analysis of Golden Rule
 - As we consider alternative values of s , $sf(k)$ pivots upward or downward, causing k^* and y^* to take on higher or lower values
 - At any steady-state k^* corresponding to a particular s , the level of c^* is measured by $(1-s)f(k^*)$, which is the vertical gap between $f(k^*)$ and $sf(k^*)$
 - As we pivot the $sf(k)$ curve by changing s , how do we make the vertical gap as large as possible?
 - This happens when the production function is parallel to the break-even line at the point where the saving curve intersects the break-even line
- Properties of Golden-Rule path
 - Slope of production function is $f'(k)$ and slope of break-even line is $n + g + \delta$, so mathematical condition for Golden Rule growth path is: Set s so that

$$f'(k^*_{GR}) = n + g + \delta$$
 - In capital-market equilibrium (which we'll study later on), profit-maximizing firms want to hire capital up to the point where the “net marginal product” of capital ($MPK - \delta$) equals the interest rate, so $r + \delta = f'(k)$
 - Thus, on Golden-Rule path: $r = n + g$, or the real interest rate equals the real growth rate of the economy.
 - If $r < n + g$, then the economy is to the right of the GR path (has higher k^* than GR) and is “overinvesting” in capital: the rate of return on capital has been driven down too low and everyone (current and future generations) would be better off with a lower s . Such an economy is *dynamically inefficient*.
 - If $r > n + g$, then economy is to the left of GR path (has lower k^*). In this case, steady-state k^* falls below the level that maximizes c^* . The

economy is not saving enough to provide the best possible living standard in perpetuity.

- It is, however, enjoying a higher level of consumption now than it would on the Golden Rule path: increasing the saving rate would lower the consumption of current people but raise that of future generations in the steady state.
- Because some people are better off and others worse off under this situation, we do not know if it is efficient or inefficient.
 - It depends on how family dynasties value current vs. future consumption.
 - We need an explicit intertemporal utility function to determine the efficient level of saving
 - We will do this in Romer's Chapter 2 in the Ramsey growth model.

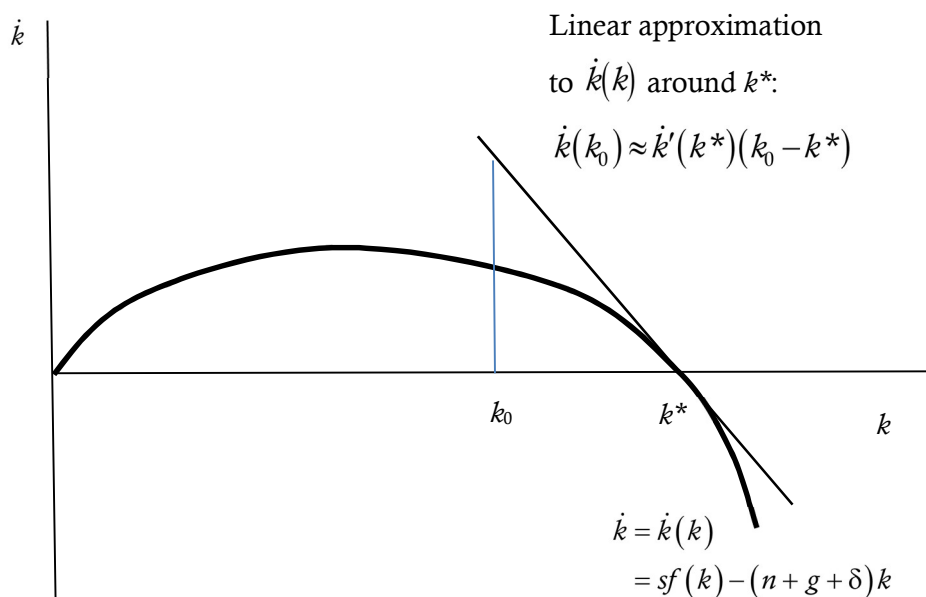
Convergence in the Solow Model

We now know how the Solow economy behaves when it is on its steady-state growth path and have shown that it will in some manner “converge” to this path. What form will this convergence take and how long will it take?

We use the second form of equilibrium graph above.

Approximating the convergence function

- We don't know the functional form of f so we can't evaluate \dot{k} corresponding to k_0 directly.
- We can approximate $\dot{k}(k_0)$ using a first-order Taylor series approximation around the steady state:
 - What line (first-order polynomial) would give the best approximation of the unknown function in a small neighborhood of k^* ?
 - Should pass through the true value of \dot{k} at $k = k^*$, which is zero.
 - Should have same slope as the true function $\dot{k}(k^*)$ at $k = k^*$, which is $sf'(k^*) - (n + g + \delta)$
 - This is the first-order Taylor approximation in a neighborhood of k^*
 - You can see from graph that approximation is pretty good very close to k^* but not so good further away (if function has strong curvature)



- Slope of tangent line is $sf'(k^*) - (n + g + \delta)$, which we can torture algebraically to get something somewhat intuitive:
 - We know that at the steady-state k , $sf(k^*) = (n + g + \delta)k^*$, so $s = \frac{(n + g + \delta)k^*}{f(k^*)}$.
 - The slope of the tangent is

$$\begin{aligned} \left. \frac{\partial \dot{k}}{\partial k} \right|_{k=k^*} &= sf'(k^*) - (n + g + \delta) \\ &= \frac{(n + g + \delta)k^*}{f(k^*)} f'(k^*) - (n + g + \delta) \\ &= \left[\frac{f'(k^*)k^*}{f(k^*)} - 1 \right] (n + g + \delta) \\ &= (\alpha_K - 1)(n + g + \delta) = -(1 - \alpha_K)(n + g + \delta) \\ &\equiv -\lambda. \end{aligned}$$
 - α_K is “capital’s share” if owners of capital are paid its marginal product:
 - f' is payment to each unit of capital
 - k^* is number of units of capital (per AL)
 - Numerator is total payments to capital
 - f is total output = total payments to all factors in economy
 - Ratio is share of total factor payments that go to capital
 - Empirically, α_K is about 1/3.
- Our approximation is the linear, first-order differential equation $\dot{k}(t) = -\lambda[k(t) - k^*]$
 - Solution to this equation (which you don’t need to know) involves a time path for k from a given initial value $k(0)$.
 - In this case, the solution is $k(t) - k^* = e^{-\lambda t} [k(0) - k^*]$.
 - The bracketed term $[k(0) - k^*]$ is the initial gap between k and k^*
 - The solution equation says that after t periods, the remaining gap between actual and steady-state k will be $e^{-\lambda t}$ share of the original gap at time 0.
 - Same convergence process applies to y as to k , so y also converges at exponential rate λ .

Convergence implications of model

- What is the pattern of convergence?
 - Asymptotic exponential convergence with a given “half-life” for the gap between actual and steady state
 - Never actually reach steady state, but after sufficient time we are arbitrarily close

- Show graph of convergence to linear path for $\log y$
- How long does convergence take?
 - To determine the half-life, set $e^{-\lambda t} = e^{-(1-\alpha_K)(n+g+\delta)t} = 1/2$ and solve for t .
 - Calibration: Romer argues that $n + g + \delta \approx 6\%$, $\alpha_K \approx 1/3$, so $\lambda \approx 0.04$.
 - $e^{-0.04t} = 1/2 \Rightarrow t \approx 18$, so the Solow model (with this calibration) predicts:
 - The economy moves about 4% of the way toward the steady-state path each year
 - It takes about 18 years to eliminate half of the gap between actual and steady-state per-capita income
- Empirical evidence
 - There have been dozens of studies of convergence across countries
 - Most of the evidence suggests that there is convergence, *if* one controls for variations in such variables as the saving rate that might affect the level of a country's steady-state path.
 - Estimated convergence rates are lower than 4%, typically closer to 2% per year. (As in Barro/Sala-i-Martin paper of the week)
 - The 2% convergence rate would fit the calibration of the model if α_K were 2/3 instead of 1/3.
 - Mankiw, Romer, and Weil (in one of the papers of the week) suggest that this is plausible if one considers human capital as part of K rather than L .

Natural Resources in the Solow Model (Save for problem set)

What are the implications of “sustainability” of natural-resource use for the Solow model? Romer shows us a stylized model with finite resources, but the conclusions depend crucially on some fairly arbitrary assumptions.

Production with land and natural resources

- Let $R(t)$ be the amount depletable natural resources used up in production at time t
- Let $T(t)$ be the amount of land used at time t
- This model is difficult to solve with general functional form, so we assume that the production function is Cobb-Douglas: $Y(t) = K(t)^\alpha R(t)^\beta T(t)^\gamma [A(t)L(t)]^{1-\alpha-\beta-\gamma}$
 - Constant returns to scale imposed
 - All exponents assumed positive
 - **Note:** Cobb-Douglas assumption is not innocuous
 - No production is possible without R or T
 - Any amount of Y can be produced with arbitrarily small amounts of R and T if only the levels of K and L are large enough
 - Elasticity of substitution among factors is assumed to be one
 - These assumptions may be valid, but it is important to recognize that we have (implicitly) made them when we choose the Cobb-Douglas form

Equations of motion for resources

- $\dot{T}(t) = 0$: Land is fixed
- $\frac{\dot{R}(t)}{R(t)} = -b < 0$
 - If the rate of use of natural resource is constant (or growing) over time, we will eventually run out.
 - The only possible steady state is with natural resource use declining sufficiently rapidly that we do not run out.
- As in the standard Solow model:
 - Labor grows at n
 - Technological progress at g
 - Constant saving rate s leads to $\frac{\dot{K}(t)}{K(t)} = s \frac{Y(t)}{K(t)} - \delta$

Steady-state analysis

- Can we find a variable like $k \equiv K / AL$ that will be constant in steady state?
 - Will there be “balanced growth” in the same sense as the usual Solow model?
 - All factors *cannot* grow at same rate, because
 - AL grows at $n + g$
 - T grows at 0
 - R grows at $-b$
 - K grows at some as-yet-undetermined rate
- Alternative strategy for finding steady-state equilibrium: *constant growth rates*
 - Search for an equilibrium in which K is growing at a constant rate
 - Follow process here that will work for many models
 - From above, $\frac{\dot{K}(t)}{K(t)} = s \frac{Y(t)}{K(t)} - \delta$
 - If $\frac{\dot{K}(t)}{K(t)}$ is to be constant over time on a steady-state growth path, then the right-hand side must be constant, and since s and δ are constant that means that $\frac{Y(t)}{K(t)}$ must be constant in the steady state.
 - If $\frac{Y(t)}{K(t)}$ is to be constant on the steady-state growth path, then Y must be growing at the same rate as K in the steady state.
 - Let's call that common steady-state growth rate g_Y^* .
 - Using the Cobb-Douglas production function (this is why we need that assumption), we can use our rules of growth rates of products and powers to get $\frac{\dot{Y}}{Y} = \alpha \frac{\dot{K}}{K} + \beta \frac{\dot{R}}{R} + \gamma \frac{\dot{T}}{T} + (1 - \alpha - \beta - \gamma) \left(\frac{\dot{A}}{A} + \frac{\dot{L}}{L} \right)$, which holds at every moment.
 - In the steady state, $\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = g_Y^*$, so $g_Y^* = \alpha g_Y^* - \beta b + (1 - \alpha - \beta - \gamma)(g + n)$ or $g_Y^* = \frac{(1 - \alpha - \beta - \gamma)(g + n) - \beta b}{1 - \alpha}$.
 - Romer shows that under reasonable conditions this steady-state growth path is unique and stable.

Implications of model

- Growth rate of per-capita income is $g_{Y/L}^* = g - \frac{\beta b + (\beta + \gamma)(g + n)}{1 - \alpha} < g$

- Thus there is a “growth drag” introduced by the presence of natural resources and land in the production function.
 - Growth drag term is zero if $\beta = \gamma = 0$ (which gets us back to a Cobb-Douglas version of the basic Solow model).
 - Romer cites Nordhaus estimates that the growth drag may be $\sim 0.25\%$ per year.
- Romer notes that Cobb-Douglas assumption that elasticity of substitution among factors is one may not be correct.
 - If elasticity of substitution is bigger than one, then it is relatively easy to substitute among inputs and the growth drag will be smaller: it is easier to get along without R and T as they become scarce relative to K and L .
 - If elasticity of substitution is less than one, then it is difficult to substitute and the growth drag will be more severe.
 - Some empirical evidence suggests elasticities of substitution > 1 , so growth drag may not (in neighborhood of today’s equilibrium) be as large as Nordhaus’s estimates.

Summing Up the Solow Model

- Simple model of evolution of capacity output in an economy with constant (exogenous) saving rate.
- There are unique, stable steady-state growth paths for K , Y , Y/L to which the economy converges asymptotically at an approximately exponential rate.
- Output grows in the steady state through (exogenous) increases in labor force and productivity ($g + n$).
- Output per worker grows at the rate of productivity growth g .
- Changes in the saving rate have level, not growth, effects on the steady-state growth path.
- Steady-state consumption per worker is maximized on the Golden Rule growth path where $r = n + g$, but this may or may not be optimal depending on how one weights current vs. future well-being.
- Introducing depletable natural resources into the model puts a drag on steady-state growth in a Cobb-Douglas version of the model.
- One crucial weakness of the Solow model is the ad-hoc assumption of a constant saving rate.
 - Is this consistent with optimal, utility-maximizing consumer behavior?
 - If we had a utility function, we could model optimal consumption and saving behavior and also perform welfare analysis balancing present vs. future.
 - That is the role of the Ramsey model, to which we now turn.