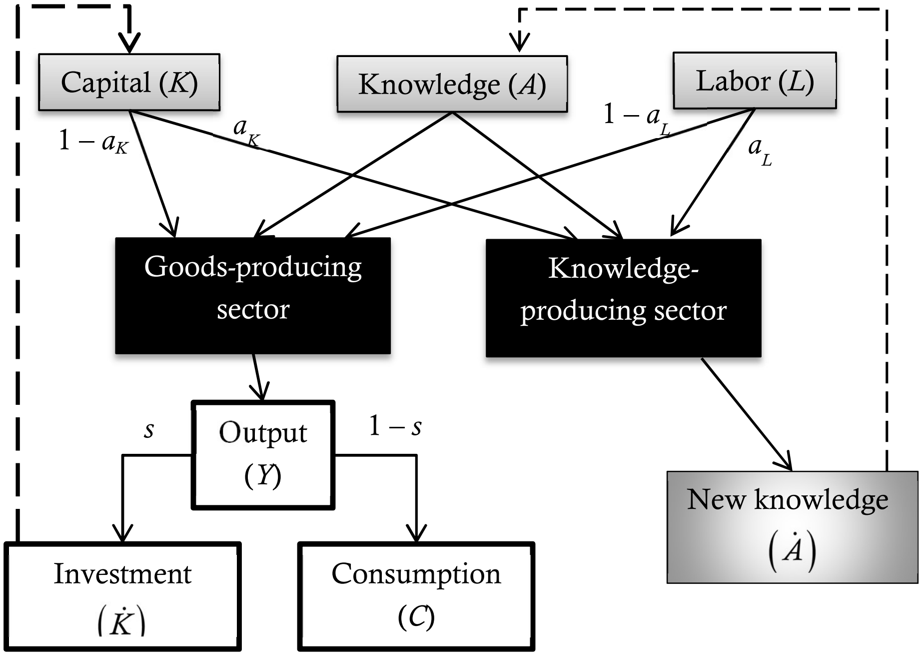
Introduction to Endogenous Growth Models

- Paul Romer's 1986 model and Robert Lucas's (1988) human capital model.
- These models get around the diminishing marginal returns to "capital" assumption by broadening the definition of capital to include knowledge or human capital, both of which may have positive externalities.
- We need some kind of external effects in order to have a model in which
  - individual firms do not have increasing overall returns to scale, so they do not expand infinitely and become economy-wide monopolies
  - the economy as a whole has increasing returns to scale, so that returns to "capital" or "produced inputs" can be constant
- Endogenous growth model have several features that economists have found attractive
  - They endogenize key parameters of the model such as g
  - They can explain lack of convergence
  - They allow s and related policy variables to affect the growth rate of GDP, not just the level of the growth path
- Text book begins with a simplified model of knowledge production via research and development in Chapter 3.
  - Uses constant saving assumption as in Solow model
  - Incorporating Ramsey saving model does not change basic dynamics
- Key characteristic leading to endogenous growth: constant returns to scale in produced inputs.
  - In Solow and Ramsey models, capital was only produced input and had diminishing returns

David Romer's R&D model

Dynamics and behavioral assumptions

- Economy has two sectors: goods-producing sector and R&D (knowledge-producing) sector
- Each sector uses labor and capital
  - a_L and a_K are the shares of labor and capital allocated to the knowledge sector
  - These should be determined by choices of owners of labor and capital allocating them to their highest return
  - Romer simplifies the model by taking these to be exogenous
  - Econ 454 studies models in which the rewards to capital and labor in the two sectors are explicitly modeled and these decisions are allowed to be endogenous



- We assume a Cobb-Douglas CRTS production function for goods:
  $Y(t) = [(1 - a_K)K(t)]^{\alpha}[A(t)(1 - a_L)L(t)]^{1-\alpha}$
- Knowledge is produced according to a Cobb-Douglas that may or may not have CTRS:
  $\dot{A}(t) = B[a_K K(t)]^{\beta}[a_L L(t)]^{\gamma} A(t)^{\theta}$
  - Note that θ = 1 ⇒ $\frac{\dot{A}}{A} = B[a_K K]^{\beta}[a_L L]^{\gamma}$ which means growth rate of *A* (our old *g*) depends on the amounts of *K* and *L* devoted to research and is constant if those amounts are constant.
  - Replication argument cannot be used to justify CRTS here
    - Same knowledge produced by two people is not twice as valuable
    - Positive spillovers could yield increasing returns to scale
    - Are other discoveries substitutes or complements for the next discovery?
- No depreciation and constant saving rate mean $\dot{K}(t) = sY(t)$
- Exogenous growth of labor force: $\dot{L}(t) = nL(t)$

Analysis of R&D Model

- Romer begins with a model in which there is no physical capital (α = β = 0)
  - We won't analyze this model in detail, but note the equations of the model if θ = 1 and *n* = 0 (so *L* is constant)

$Y(t) = A(t)(1-a_L)L$

  - $\frac{\dot{A}(t)}{A(t)} = B[a_L L]^\gamma$
  - Output is proportional to A and the growth rate of A is a constant, so this model has a constant growth rate of output that is determined by B, a_L, and L (and γ).
  - Higher R&D productivity, more labor being used in the labs, and a bigger population all lead to a higher growth rate (not just to a higher, parallel growth path)

- For the full model (with K), we have two state variables, A and K
  - We denote the growth rates of A and K by g_A and g_K
- **Dynamics of *K***

$$\dot{K}(t) = sY(t) = \left[s(1-a_k)^\alpha (1-a_L)^{1-\alpha}\right] K(t)^\alpha A(t)^{1-\alpha} L(t)^{1-\alpha}$$

  - The term in brackets is a constant (over time) that we shall call c_K
  - The growth rate of K at every moment t is

$$g_K(t) \equiv \frac{\dot{K}(t)}{K(t)} = c_K \left[\frac{A(t)L(t)}{K(t)}\right]^{1-\alpha}$$

  - We seek a steady state in which K grows at a constant rate g_K^*, so we want to analyze the change in or growth rate of the growth rate
  - $\frac{\dot{g}_K(t)}{g_K(t)} = (1-\alpha)(g_A + n - g_K)$ using our rules for growth rates
    - $$\left.\begin{array}{l} \dot{g}_K(t) = 0 \text{ if } g_K(t) = g_A(t) + n \\ \dot{g}_K(t) > 0 \text{ if } g_K(t) < g_A(t) + n \\ \dot{g}_K(t) < 0 \text{ if } g_K(t) > g_A(t) + n \end{array}\right\} \Rightarrow \dot{g}_K = 0 : g_K^* = g_A^* + n$$
    - The $\dot{g}_K = 0$ curve is a line with slope of one and intercept on the g_K axis at $n \geq 0$.
      - Below the line, $\dot{g}_K > 0$ and above the line $\dot{g}_K < 0$, so the arrows point vertically toward the line

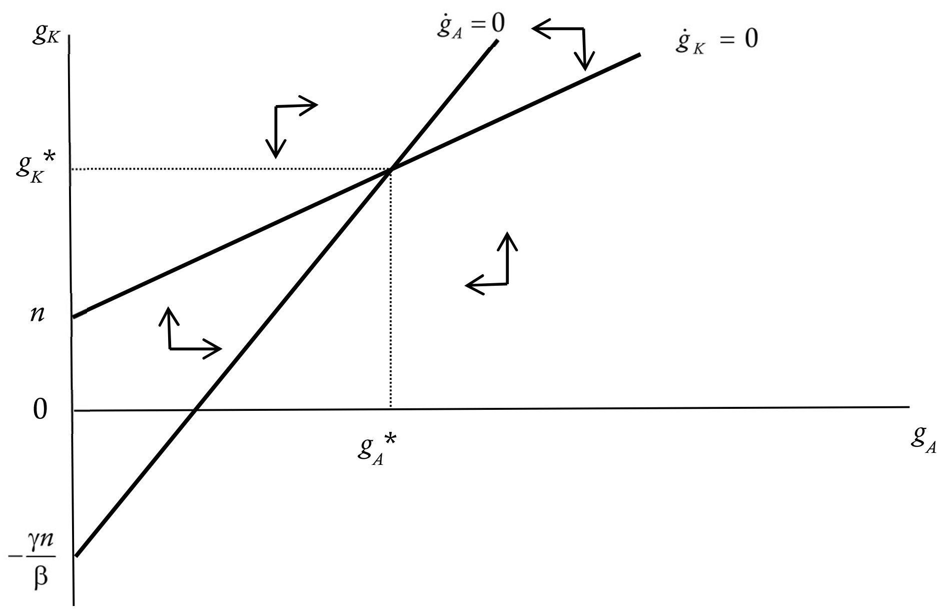
- **Dynamics of *A***

$$g_A(t) = \frac{\dot{A}(t)}{A(t)} = \left[B a_K^\beta a_L^\gamma\right] K(t)^\beta L(t)^\gamma A(t)^{\theta-1}$$

  - The term in brackets is constant over time and called c_A

- As above, the growth rate of the growth rate at every moment t is
  $$\frac{\dot{g}_A(t)}{g_A(t)} = \beta g_K(t) + \gamma n + (\theta - 1) g_A(t)$$
  - $$\left.\begin{array}{l} \dot{g}_A(t) = 0 \text{ if } g_K(t) = -\frac{\gamma n}{\beta} + \frac{1-\theta}{\beta} g_A(t) \\ \dot{g}_A(t) > 0 \text{ if } g_K(t) > -\frac{\gamma n}{\beta} + \frac{1-\theta}{\beta} g_A(t) \\ \dot{g}_A(t) < 0 \text{ if } g_K(t) < -\frac{\gamma n}{\beta} + \frac{1-\theta}{\beta} g_A(t) \end{array}\right\} \Rightarrow \dot{g}_A = 0 : g_K^* = -\frac{\gamma n}{\beta} + \frac{1-\theta}{\beta} g_A^*(t)$$
  - The $\dot{g}_A = 0$ curve is a line with slope $\frac{1-\theta}{\beta}$ and intercept on the vertical (g_K) axis at $-\gamma n/\beta \le 0$.
  - To the left of the line $\dot{g}_A > 0$ and to the right of the line $\dot{g}_A < 0$, so the arrows point horizontally toward the line
- **Equilibrium dynamics**
  - The nature of the equilibrium depends crucially on two properties of the parameters:
    - $n > 0$ vs. $n = 0$
      - This determines whether there is any exogenous source of growth in the model
      - If $n > 0$ as in the Solow and Ramsey models, sustained growth in total GDP is possible through exogenous growth in L
    - $\theta + \beta = 1$ vs. $\theta + \beta < 1$ (or $\theta + \beta > 1$)
      - This determines "returns to scale in produced inputs"
      - Note that K and A are "produced" in the model
      - The production function for goods always has constant returns in produced inputs because K has exponent α and A has exponent $1 - \alpha$
      - The production function for knowledge has returns to scale in the two produced inputs equal to the sum of their exponents: $\theta + \beta$
      - If $\theta + \beta = 1$, then the model can sustain ongoing "endogenous" growth even if $n = 0$ because increases in both K and A together are not subject to diminishing returns
- **Dynamics with $n > 0$ and $n = 0$**
  - With $n > 0$, the $\dot{g}_K = 0$ line intercept is positive and the $\dot{g}_A = 0$ line intercept is negative
  - If $n = 0$, both lines pass through the origin
  - **Case I: $\beta + \theta < 1$** (diminishing returns in produced inputs)

- Slope of $\dot{g}_A = 0$ line is $\frac{1-\theta}{\beta} > 1$, so it is steeper than the $\dot{g}_K = 0$ line



- Economy converges to unique equilibrium from all points in space
- Solving algebraically, we can show

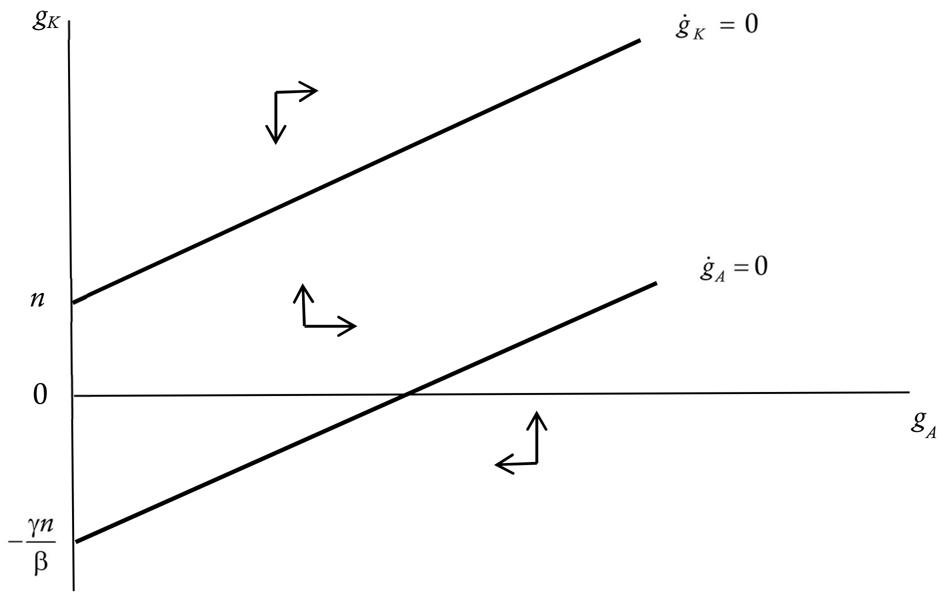
$$g_A^* = \frac{\beta + \gamma}{1 - (\beta + \theta)} n$$

$$g_K^* = \frac{1 - \theta + \gamma}{1 - (\beta + \theta)} n$$

- Growth here is exogenous in the sense that if $n = 0$, both K and A stop growing. (Note that both lines intercept at the origin if $n = 0$.)
- This case replicates the dynamics of the Solow model with g determined endogenously as a function of n

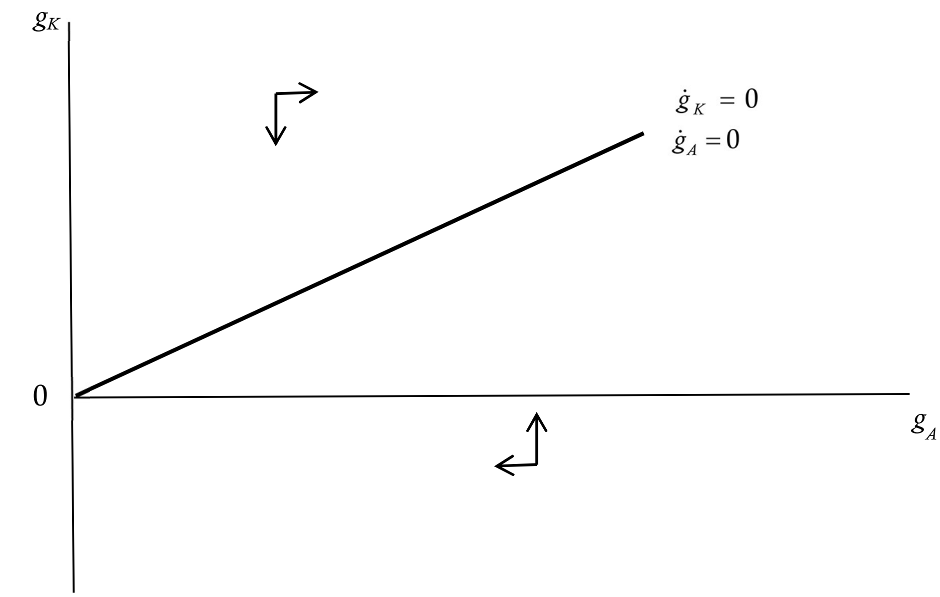
o **Case II: β + θ = 1**

- In this case, the slope of the $\dot{g}_A = 0$ line $\frac{1-\theta}{\beta} = 1$ and the two lines are parallel (or coincident)
- If $n > 0$, then they are parallel



- The economy will move into the channel between the lines and then growth in both *K* and *A* will accelerate forever.
- Intuitively, a bigger economy means more scientists means more discoveries means faster growth. As long as $n > 0$, the exogenous growth in the labor force leads to accelerating growth.

▪ If $n = 0$, then the two lines coincide



- Economy converged to the line and on the line, both growth rates are constant and equal (because the line has slope of one)

- Because $g_K^* = g_A^*$, K/A is constant in the steady state
    - There is a unique K/A^* that will sustain equal growth in K and A and a unique common growth rate g^* that is consistent with that K/A^*
    - You will work out the algebra in Problem 3.5.
- Examples of this case: Suppose that $B\uparrow$ so that c_A increases.
    - This raises g_A and moves the economy to a point to the right of the original equilibrium.
    - Economy converges back up and to the left to a new equilibrium that is higher than original.
- Second example: $s\uparrow$ so that c_K increases
    - Raises g_K and moves upward above original equilibrium
    - Economy converges down to the right to a new high growth rate
- Third example: $a_K\uparrow$ so that c_K falls and c_A increases
    - Economy moves down and to the right
    - Converges back to line, but could be higher or lower growth rate
    - Change in growth rate depends on the productivity of A vs. K at the margin.
- Endogenous growth occurs in this case: economy sustains positive growth even when there is no exogenous source (n = 0)
- Growth rate depends (positively) on s, B, a_K, a_L, and L

- **Case III: β + θ > 1**
    - In this case, the $\dot{g}_A = 0$ line is flatter than the $\dot{g}_K = 0$ because $\frac{1-\theta}{\beta} < 1$
    - This case looks like Case II, but the lines are not parallel.
    - In this case, we get explosive growth even when n = 0.

Microeconomics of R&D

- The key question that we have dodged in Romer's R&D model: What determines a_K?
    - Capital owners must decide whether to build factories or labs
    - Economists would assume that the choose the use of their capital that provides the higher rate of return
        - So in equilibrium the amount of capital in the two sectors would have to balance the marginal rates of return

  - Rate of return on factories is straightforward: They produce output that is sold to earn revenue
- How do labs earn money?
  - In the real world, there are lots of funding sources for R&D
    - Corporate funding
    - Government grants
    - Tuition from university students
    - Since we don't model government or university research, we are interested mostly in corporate-funded research and development
  - In our model, knowledge is purely non-rival and non-excludable
    - Any discovery is immediately useful to all producers
    - There is no "appropriability" of knowledge for private benefit
    - New knowledge cannot be sold or used profitably
    - Why would capital owners put money into labs that earn nothing?
      - They wouldn't, so we would need to build a model of how lab owners can earn money from R&D in order to pay for the capital and labor that is used.
- Models of a_K
  - Corporate R&D is profitable if there is an effective way for the company to appropriate the knowledge
  - This usually occurs by preventing other firms from using the knowledge created through some kind of "appropriability mechanism"
    - May also involve licensing
    - Note that either is inefficient, because once created the knowledge is nonrival and "should" be universally used for free
  - Two common appropriability mechanisms are intellectual property rights (patents) and secrecy
    - Both are flawed
    - Some kinds of intellectual property are better protected by patents, some by secrecy, and others are virtually unprotectable
  - Effective patent protection or secrecy gives an effective (but usually temporary) monopoly on the use of the knowledge to the firm doing the R&D
  - Two common models for a_K are based on this:
    - A model of product innovation in which R&D can produce new varieties of (intermediate) goods on which the innovating firm holds a monopoly
    - A model of process innovation in which R&D can advance productive efficiency of one (intermediate) good (of many) and have a cost advantage in production until another firm leap-frogs it
    - Both models add complexity to Romer's R&D model because both require multiple goods in order to have more than one firm

- We study both models in Econ 454

Model of Learning by Doing

- Romer's short section on learning by doing develops the essence of Paul Romer's first (1986) endogenous growth model.
  - Kenneth Arrow developed a model in the 1960s based on the idea that a firm's A would be increased as it produced output, so $\dot{A} \sim Y$
  - Paul Romer's version of this was slightly different
    - Firms' learning is related to capital accumulation rather than output
    - Knowledge is non-appropriable
      - New knowledge occurs as a by-product of capital investment
      - Firms have (some) incentive to invest, so knowledge creation happens despite pure nonrivalry
- Learning by doing with a **constant saving rate**
  - $$Y(t) = K(t)^{\alpha}\left[A(t)L(t)\right]^{1-\alpha}$$ $$A(t) = BK(t)^{\phi}$$
  - Solving out A yields $$Y(t) = K(t)^{\alpha} B^{1-\alpha} K(t)^{\phi(1-\alpha)} L(t)^{1-\alpha}$$ $$\dot{K}(t) = sY(t) = sB^{1-\alpha} L(t)^{1-\alpha} K(t)^{\alpha+\phi(1-\alpha)}$$
  - This model converges, has endogenous growth, or explodes as $\phi < 1$, $\phi = 1$, $\phi > 1$
  - Case of $\phi = 1$ is the endogenous-growth case
    - Let $n = 0$ so there is no exogenous growth
    - $$Y(t) = (BL)^{1-\alpha} K(t) \equiv bK(t)$$ $$\dot{K}(t) = sY(t) = sbK(t)$$
    - $$\frac{\dot{K}(t)}{K(t)} = sb$$
    - Thus, growth in the capital stock and output is constant at rate sb
    - Any increase in saving, in the productivity of learning, or in the labor force would increase growth
- **Ramsey consumers** in the learning-by-doing model (not done this way in 4^{th} edition)
  - Assume $\phi = 1$ and $n = 0$, so we have the endogenous-growth case
  - Aggregate knowledge is proportional to aggregate capital stock (but this is not the case at the firm level)
    - Firms take aggregate knowledge as given and do not consider how their own investment will add to it because they are small

$Y_i(t) = K_i(t)^{\alpha}\left[A(t)L_i(t)\right]^{1-\alpha}$

- $A(t) = BK(t)$

  $Y_i(t) = B^{1-\alpha}K(t)^{1-\alpha}\,K_i(t)^{\alpha}\,L_i(t)^{1-\alpha}$

- The private marginal product of capital is

  $\dfrac{\partial Y_i(t)}{\partial K_i(t)} = \alpha B^{1-\alpha}K(t)^{1-\alpha}\left[\dfrac{K_i(t)}{L_i(t)}\right]^{-(1-\alpha)} = r(t)$ (there is no depreciation)

  - Each firm sets its $\dfrac{K_i}{L_i}$ so that the private marginal product equals the economy-wide interest rate r
  - This means that all firms have the same $\dfrac{K_i}{L_i}$ that is equal to the aggregate K/L
  - Setting $K_i/L_i = K/L$,

    $$r(t) = \alpha B^{1-\alpha}K(t)^{1-\alpha}\left[\frac{K(t)}{L(t)}\right]^{-(1-\alpha)}$$

    $$= \alpha B^{1-\alpha}K(t)^{1-\alpha}\,K(t)^{-(1-\alpha)}\,L(t)^{1-\alpha}$$

    $$= \alpha\left[BL(t)\right]^{1-\alpha} = \alpha\left[BL\right]^{1-\alpha} = \bar{r}$$

  - This rate of return $\bar{r}$ is constant over time (with $n = 0$) and depends on the rate of knowledge accumulation through investment B, α, and the size of the labor force
  - Note that the marginal *social* product of capital (varying K as well as K_i) is larger than the marginal *private* product
    - $MSP_K = \left.\dfrac{\partial Y_i}{\partial K_i}\right|_{K_i = K} = (BL)^{1-\alpha} > MPP_K$
    - This means that individual firms will underinvest in capital
    - They do not take into account the positive social externality that their investment conveys on all firms through increased knowledge
    - This means that the privately generated growth rate will be lower than the socially optimal growth rate

- Ramsey consumers, as usual, choose a consumption path that satisfies the Euler equation $\dfrac{\dot{C}(t)}{C(t)} = \dfrac{r(t)-\rho}{\theta}$.

  - In this case, $r(t) = \bar{r}$ and $\dfrac{\dot{C}(t)}{C(t)} = \dfrac{\bar{r}-\rho}{\theta} = \dfrac{\alpha(BL)^{1-\alpha}-\rho}{\theta} = \bar{g}$

- To satisfy the economy's budget constraint, Y must grow at the same rate as C, so the economy grows at $\bar{g}$ at every instant.
- There are no convergence dynamics: wherever an economy is, it just grows at $\bar{g}$ from there. (Poor countries with same parameters do not catch up.)
- Growth rate $\bar{g}$ depends on parameters of economy: α, B, L, ρ, θ.
- Once again, we have "scale effects" because a larger L means a faster growth rate.
  - If we allow $n > 0$, then we have both endogenous and exogenous growth and the growth rate accelerates over time.
  - Are scale effects realistic? Some argue no, but Kremer's argument for Eurasia, Australia, and Tasmania seems to provide some support.
  - In addition, there is much evidence that growth has accelerated over the centuries (as population has grown).
- Non-optimality: social planner would internalize the knowledge externality and use $r^* = (BL)^{1-\alpha} > \bar{r} = \alpha (BL)^{1-\alpha}$ leading to faster growth at $g^* = \frac{(BL)^{1-\alpha} - \rho}{\theta} > \bar{g}$

(Paul) Romer model (not worth doing the details)

- What's different about this model?
  - We model the incentives for production of knowledge explicitly
  - We introduce the "Ethier production function" and the now-ubiquitous model of a continuum of "intermediate goods"

Human Capital in the Solow Model

- Distinction between knowledge capital and human capital
  - Latter is rival and embodied in worker
  - Former relates to nonrival ideas that all share (costlessly)
- Model is motivated by the dominant question: "Why are some countries richer than others?"
  - Solow model says differences in k
    - Not plausible (as Romer shows late in Ch 1)
  - Mankiw, Romer, & Weil: differences in physical and human capital
    - They argue this is plausible; others disagree
  - Differences in A
    - Why would technology be different across countries?
    - Barriers (legal and otherwise) to adoption

        - Non-applicability of advanced technologies in poor countries (climate, unreliable physical infrastructure, etc.)
    - Differences in "social infrastructure"
        - We'll have more to say about this soon
- How to incorporate human capital into model?
    - Many alternative ways; Romer does one (and others in problems 4.8 and 4.9)
    - How does economy "produce" human capital?
        - Process of education or training has two major costs: teachers' time (for which they are paid) and students' time (for which they are not paid)
        - Can use a two-sector model with a production function for education using labor (teachers) and capital (schools) like the one for knowledge in the R&D model
        - Can just deduct some amount of a conglomerate "output" as being education in a one-sector model (like some output is physical capital rather than consumption). This is Romer's 4.8.
        - Can model the process as holding people out of the labor force during an education period. This is Romer's Section 4.1.
            - This doesn't model the cost of teachers and schools.
            - Note that forgone earnings may be higher than teacher/school costs at most schools (if maybe not at Reed)

Simple human-capital model setup

- Let $H(t) \equiv L(t)G(E)$ be the amount of human capital, which is the number of workers $L(t)$ times the amount of human capital per worker $G(E)$, where E is the average education level of current workers.
    - $G'(E) > 0$
    - $G(E) = e^{\phi E}$ is a commonly used functional form
    - We assume that in a steady state with education level E, people live T years, going to school for E years and working for $T - E$ years.
    - In general (but not in this model), human capital includes not just education but training, health and other "acquired" characteristics that affect labor productivity.
- $Y(t) = K(t)^{\alpha}\left[A(t)H(t)\right]^{1-\alpha}$
- $\dot{K}(t) = sY(t) - \delta K(t)$
- $\dot{A}(t) = gA(t)$
- $\dot{L}(t) = nL(t)$

Solving the model

- This model looks (and behaves) similarly to Solow model
- Define $k \equiv \frac{K}{AH} = \frac{K}{ALG(E)}$
- $\dot{k}(t) = sf(k(t)) - (n+g+\delta)k(t)$
- $\dot{k} = 0 \Rightarrow k = k^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$
- How will a **change in *E*** affect the steady-state growth path?
  - Effects of $E\uparrow$ (or $G\uparrow$) on K and Y are equivalent to increase in L
  - Economy moves to higher, parallel steady-state path
  - Level effect, but no growth effect
  - Y and Y/L are higher in steady-state
- But the important variable (living standards) here is Y/N, where N is total population
  - $\left(\frac{Y(t)}{N(t)}\right)^* = y^* A(t) G(E) \left(\frac{L(t)}{N(t)}\right)^*$ on the steady-state path
    - Increase in E does not affect y^* or $A(t)$
    - Increase in E raises $G(E)$
    - Increase in E lowers L/N because more people are in school and fewer in the labor force
    - What will be the net effect?
  - What is L/N?
    - It seems like it should be $(T-E)/T$ since that is the ratio of working years to total life years for each individual
    - That is correct if $n = 0$
    - If the population is growing, then the cohort in education is larger than the cohort that is working.
    - Romer (and Coursebook) shows that in steady state $\frac{L(t)}{N(t)} = \frac{e^{-nE} - e^{-nT}}{1-e^{-nT}}$
    - It is intuitively clear (and mathematically easy) that $\frac{\partial (L/N)}{\partial E} < 0$

Dynamics of increase in E

- Initial effect lowers Y because fewer people in labor force but no immediate increase in the education of those who are working

- In steady state, the two effects noted above are in conflict and we don't know which will dominate
  - $\frac{\partial(Y/N)}{\partial E} = \frac{\partial(Y/N)}{\partial E} + \frac{\partial(Y/N)}{\partial(L/N)}\frac{\partial(L/N)}{\partial E}$
  - The first term depends mostly on $G'(E)$ and the second is negative.
  - If $G'(E)$ is large, then Y/N is likely to rise with an increase in E
  - This makes intuitive sense: if education is highly productive it will raise per-capita income; if it is not, then it drains people who could be working into useless education.