The Diamond Model: Overlapping Generations

Differences from the Ramsey model

- Discrete time
- Finite (two-period) lifetimes
- Distinct phases of life (work and retirement)

Similarities to Ramsey model

- Same basic idea of consumption/saving
- Same production and growth dynamics

Dynamic assumptions

- Lifetime assumptions
    - Individual lives two periods
    - No links to earlier or later generations
    - Works first period and lives off saving (with interest) second period
    - Old gradually sell off capital to the young throughout the old-age period
    - Young save by buying capital from old, then diverting output from consumption to create additional new capital as desired
- Size of population
    - L_t = number of young in year t
        - They are the only workers
    - $L_{t+1} = (1+n)L_t$
        - Labor force and population grow at annually compounded rate n
- Consumption notation
    - $C_{1,t} =$ consumption per person by young in period t
    - $C_{2,t} =$ consumption per person by old in period t

Utility

- Period felicity function is again CRRA
- Consumption choices by person who is young in t
    - Consumes $C_{1,t}$ when young and $C_{2,t+1}$ when old
- Utility is $U = \frac{C_{1,t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2,t+1}^{1-\theta}}{1-\theta}$, with $\theta > 0$ and $\rho > 0$.
    - Utility is discounted at annually compounded rate ρ

Production and dynamics

- $Y_t = F\left(K_t, A_t L_t\right)$ with constant returns to scale and usual marginal product conditions
  - $y_t = f\left(k_t\right)$ as before
  - $r_t = f'\left(k_t\right)$ (assume no depreciation)
  - $w_t = f\left(k_t\right) - k_t f'\left(k_t\right)$ = wage of each effective labor unit
    - $A_t w_t$ = wage per worker
    - $A_t w_t L_t =$ total wages earned in economy (= income of young)
- All capital is owned by the old at the beginning of each period
- $K_{t+1} = L_t\left(w_t A_t - C_{1,t}\right)$
  - Expression in parentheses is saving by each young person, which consists of buying up capital from oldies plus perhaps diverting some new production to further capital.
  - Multiply by number of young people to get total capital put away for next period

Budget constraint and utility maximization

- Budget constraint over lifetime is $C_{1,t} + \dfrac{C_{2,t+1}}{1+r_{t+1}} = A_t w_t$
  - Or $C_{2,t+1} = \left(1+r_{t+1}\right)\left(w_t A_t - C_{1,t}\right)$
- Individual maximization problem:

$$\max_{C_{1,t}, C_{2,t+1}} \left[\frac{C_{1,t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2,t+1}^{1-\theta}}{1-\theta}\right], \text{ subject to } C_{1,t} + \frac{C_{2,t+1}}{1+r_{t+1}} = A_t w_t$$

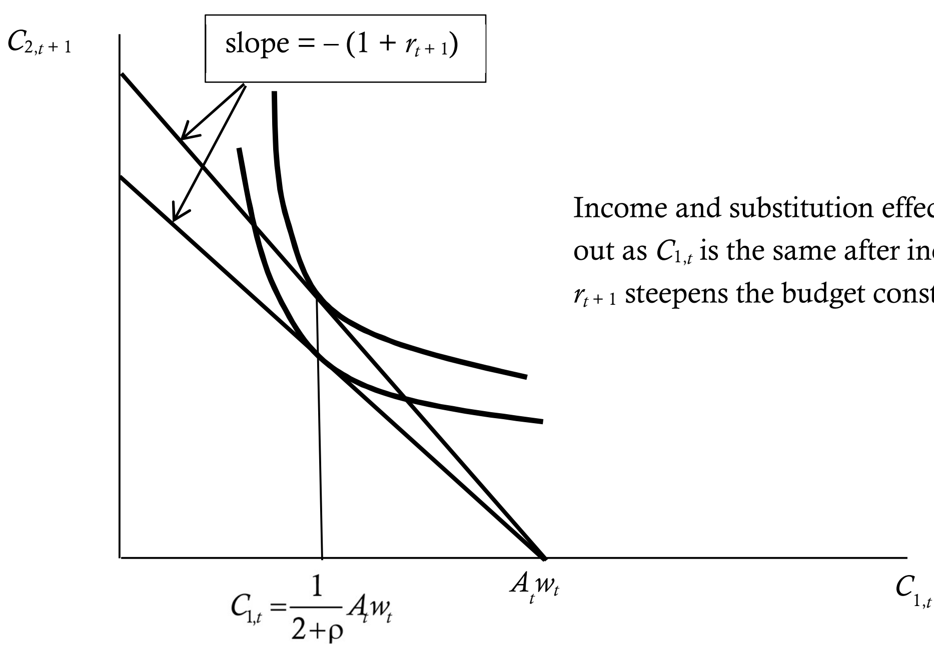
- Can do this as a Lagrangean, but just as easy to solve and substitute for $C_{2,t+1}$:
  - $$\max_{C_{1,t}} \left[\frac{C_{1,t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{\left[\left(1+r_{t+1}\right)\left(w_t A_t - C_{1,t}\right)\right]^{1-\theta}}{1-\theta}\right]$$
- First-order conditions for this maximization come from $\dfrac{dU}{dC_{1,t}} = 0$
  - $$\frac{C_{2,t+1}}{C_{1,t}} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}}$$
  - Budget constraint: $C_{2,t+1} = \left(1+r_{t+1}\right)\left(w_t A_t - C_{1,t}\right)$
  - Note similarity of first condition to Euler equation in Ramsey model:
    - C increases or decreases over time as $r > \rho$ or $r < \rho$
    - Sensitivity of consumption path to r depends on $1/\theta$

- Plugging the budget constraint back into the other first-order condition yields

$$C_{1,t} = \frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} A_t w_t \equiv \left(1 - s(r_{t+1})\right) A_t w_t$$

  - Note that numerator and denominator terms are always positive and that the denominator is always larger
  - If $\rho \approx r$ then we consume about ½ of income in first period and ½ in second, which is consistent with basic consumption smoothing

- We can show that $s'(r_{t+1}) = \dfrac{(1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} > 0$ iff $\theta < 1$
  - Change in r has income and substitution effects
    - Reward to saving is higher if $r\uparrow$
    - Don't need to save as much for retirement if $r\uparrow$
  - Remember that $1/\theta$ is elasticity of intertemporal substitution
    - If $1/\theta$ is large, then substitution effect is strong and $s' > 0$
    - If $1/\theta$ is small, then income effect dominates and $s' < 0$
  - Intermediate case $\theta = 1$ has $s' = 0$ and saving rate does not depend on interest rate
    - Recall that the CRRA utility function approaches $u = \ln(c)$ as $\theta \to 1$.
    - If $\theta = 1$, then $s = \dfrac{1}{2+\rho}$ = constant
    - We shall use log utility as a special case because it is simple
    - What does this mean in terms of indifference curves?



Income and substitution effects cancel out as $C_{1,t}$ is the same after increase in r_{t+1} steepens the budget constraint.

Analysis of the Diamond Model

Dynamics

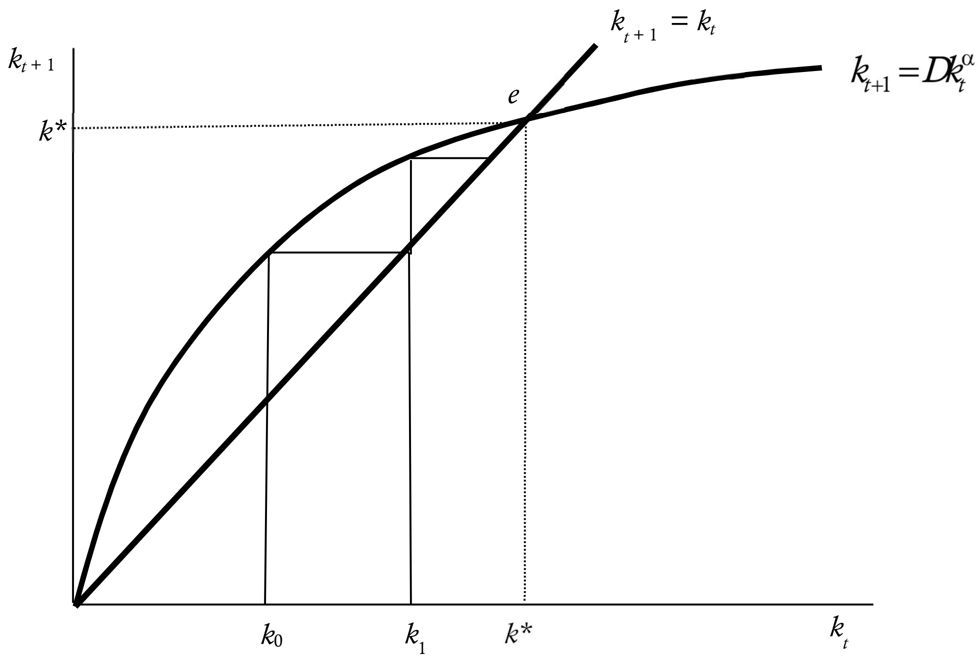
- The basic equation of motion of this model is $K_{t+1} = s(r_{t+1}) L_t A_t w_t$
- We want to translate this into k_{t+1}:

$$k_{t+1} \equiv \frac{K_{t+1}}{A_{t+1}L_{t+1}} = s(r_{t+1})\frac{A_t w_t L_t}{A_{t+1}L_{t+1}}$$

$$= s(r_{t+1})\frac{w_t}{\frac{L_{t+1}}{L_t}\cdot\frac{A_{t+1}}{A_t}} = s\left[f'(k_{t+1})\right]\frac{f(k_t) - k_t f'(k_t)}{(1+n)(1+g)}$$

  - This equation gives k_{t+1} implicitly as a function of k_t, but it can't be solved in the general case.
- **Steady-state condition**
  - What would correspond to $\dot{k} = 0$?
  - $\Delta k_{t+1} \equiv k_{t+1} - k_t = 0$ would be the equivalent in discrete time
  - Setting $\Delta k_{t+1} = 0$ (or $k_{t+1} = k_t = k^*$) gives the steady-state condition:

    $k^* = \dfrac{s\left[f'(k^*)\right]\left[f(k^*) - k^* f'(k^*)\right]}{(1+n)(1+g)}$, which implicitly defines k^* the steady-state value of k.
  - This equation is difficult to work with because we don't know the form of f and we don't even know the sign of s'.
  - Depending on the forms of s and f, the function on the right can have a variety of shapes.
- **Special case: θ = 1 (log utility) and Cobb-Douglas production function $y = k^\alpha$**
  - In this case, $s(r) = \dfrac{1}{2+\rho}$, $f'(k) = \alpha k^{\alpha-1}$, $w = (1-\alpha)k^\alpha$.
  - $k_{t+1} = \dfrac{(1-\alpha)k_t^\alpha}{(2+\rho)(1+n)(1+g)} \equiv D k_t^\alpha$ with positive constant D
  - In this case, we can graph k_{t+1} as a function of k_t and know its basic shape:



- In this case, if we start at k_0, we will converge to $k^* = D^{\frac{1}{1-\alpha}}$ by the "cobweb" path shown
- Note effects of parameters:
  - $\rho\uparrow \Rightarrow D\downarrow$
  - $n\uparrow \Rightarrow D\downarrow$
  - $g\uparrow \Rightarrow D\downarrow$
  - Effects are similar to Ramsey (and Solow) model

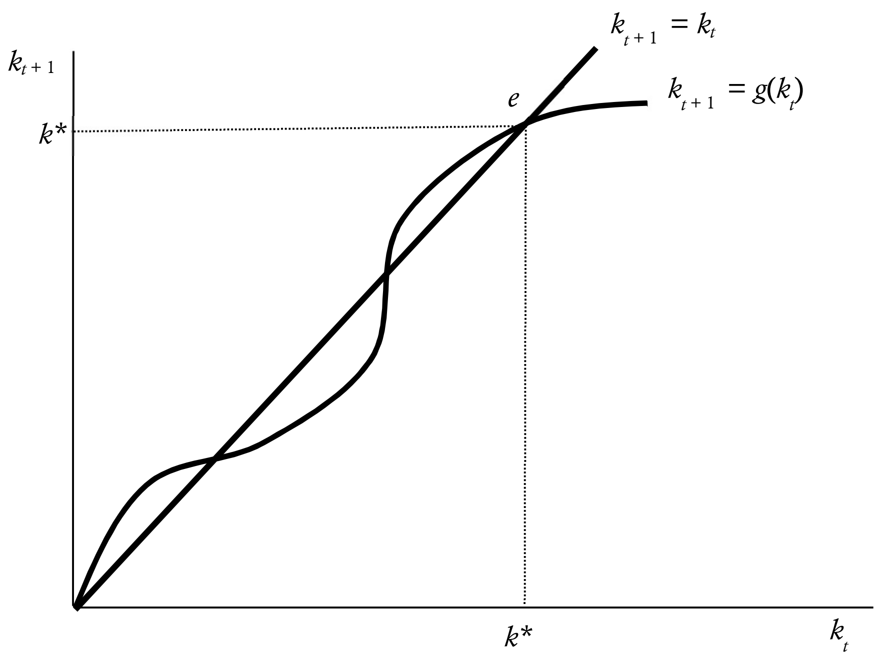
Properties of Diamond Steady State

- In steady-state equilibrium
  - k and y are stable
  - Y/L grows at g
  - Y, K grow at $n + g$
- Speed of convergence:
  - $k_{t+1} - k_t \approx \alpha\left(k_t - k^*\right)$
  - $\alpha \approx 1/3$, so economy moves 1/3 of way to equilibrium in each "period"
    - Note that "period" is half a lifetime, so this is not so different from Solow/Ramsey result

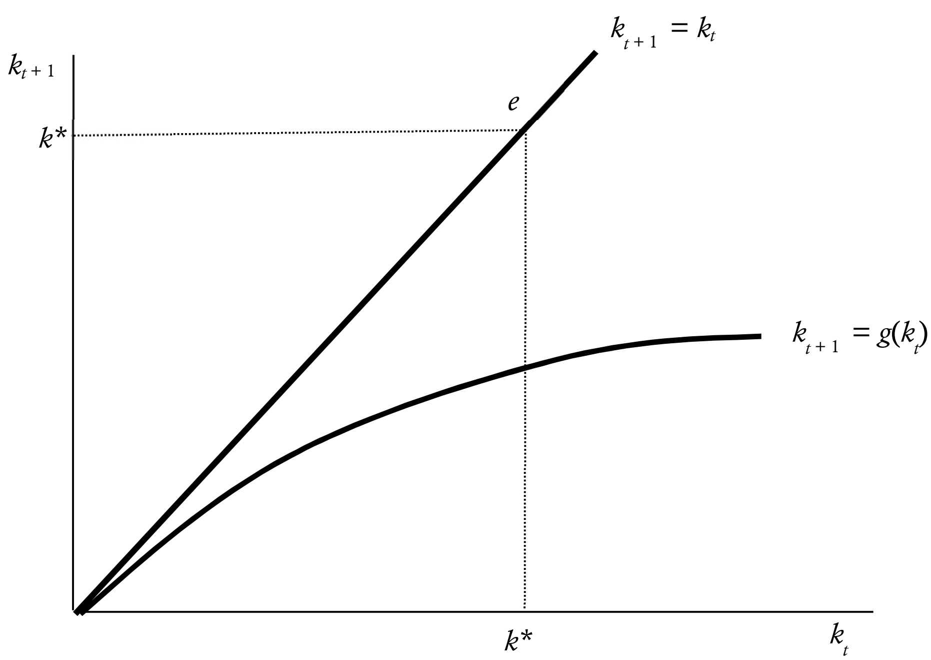
Diamond Model General Case?

- If we abandon the comfortable home of log utility and Cobb-Douglas we admit to strange possibilities:

- Can have multiple equilibria



- Can have no non-zero equilibrium at all



Dynamic Inefficiency in Diamond Model

- Equilibrium in Ramsey model was Pareto efficient
- Diamond model admits the possibility of inefficiency
  - It is possible that k^* is greater than Golden Rule k

    - Why? Because survival in old age depends on saving lots of income regardless of rate of return.
    - It is possible that the saving rate that is optimal for an individual might be higher than the saving rate that leads to Golden Rule level of k^*
- Suppose that $k^* > k_{GR}$
    - Suppose that there was another (than capital) way of transferring money from youth to old age (Social Security) so that saving could go down
    - Young would be better off because they could consume more
    - Future generations would be better off because $k^*\downarrow$ means higher steady-state c^*
    - Everyone is made better off and no one worse off, so original equilibrium must not have been Pareto optimal
- How can model by inefficient? Where is the market failure?
    - No externalities, but an absent market: no way for current generation to trade effectively with future generations
    - Only way to provide for retirement is through saving, even if rate of return is zero or negative
    - From social standpoint, it is desirable to avoid low- or negative-return investment, but for individual facing retirement this is only choice
    - If benevolent government were to establish transfer scheme from young to old (like Social Security), then they would not need to accumulate useless capital in order to eat in retirement
    - You will do a problem this week looking at the effect of alternative Social Security regimes in the Diamond model
- Is this empirically relevant?
    - $k > k_{GR}$ means that $f'(k) - \delta < g + n$
    - Are interest rates lower than the GDP growth rate?
        - Probably not in U.S. steady state