6 Stochastic Growth Models and Real Business Cycles

Chapter 6 Contents

A. Topics and Tools

Economists recognized the existence of cyclical fluctuations in economic activity long before modern macroeconomic measurements such as GDP and the unemployment rate were collected and published. Indeed, one macroeconomics text cites a reference to something analogous to a business cycle in biblical sources! During the latter half of the 19th century, economists began to note recurrent booms and depressions of the industrial economy in which each "trade cycle" resembled the others in many respects. Business-cycle analysis began in earnest in the 1890s. An early comprehensive compilation of business-cycle statistics was Wesley Clair Mitchell (1913).[1] Dennis Robertson (1915) and A.C. Pigou (1933) were among the leading economists who developed theories to try to explain Mitchell's empirical business-cycle observations.

It was the depth and persistence of the Great Depression of the 1930s that brought business-cycle analysis to the center of economists' attention. Existing theories were inadequate to explain a decline in real output of over 30 percent and an unemployment rate that rose to 25 percent and remained in double digits for a decade.

John Maynard Keynes (1936) transformed the analysis of business cycles and effectively founded the discipline of macroeconomics with his *General Theory of Employment, Interest and Money*, which modeled the behavior of an economy in severe depression. Keynes focused the attention of economists on the role of deficient demand in cyclical downturns. He believed that investment, operating through its effect on aggregate demand, was the primary engine driving the business cycle. Although developed to explain a depressed economy, Keynes's model remained the dominant macroeconomic paradigm through the early post-World-War-II period.

The dominance of Keynes's ideas began to wane in the 1970s when the combination of inflation and oil-shock-induced stagnation in production—stagflation—presented a situation that did not fit the traditional Keynesian theory. Stagflation disrupted the empirical status quo of macroeconomics at the same time that a desire to understand the microeconomic foundations of macro theory created new theoretical challenges. Macroeconomists tried to understand the new events by building models of the business cycle based on rigorously specified sets of microeconomic assumptions such as utility maximization. The new classical and new Keynesian theories that grew out of this first generation of "microfoundations" models are the subject matter of an upcoming section of this course. Robert Lucas developed a model with imperfect information and market clearing that seemed to explain some of the more prominent

[1] The third part of this work was published separately in 1941 as *Business Cycles and their Causes*, which was reprinted in a paperback edition by Porcupine Press in 1989.

business-cycle features. "New Keynesian" macroeconomists responded with an alternative set of theoretical explanations based on sticky wages or prices. Later new Keynesians emphasized the presence of coordination failures that lead to inefficiencies in aggregate equilibrium.

In this chapter, the focus is on "real business cycle" (RBC) models. These models attempt to explain the business cycle entirely within the framework of efficient, competitive market equilibrium. They are a direct extension of the Ramsey growth model. However, unlike the Ramsey model, the rate of technological progress is assumed to vary over time in response to shocks, which leads to fluctuations in the growth rate. In the initial literature their builders "calibrated" RBC models by choosing values for their behavioral parameters (such things as ρ and θ from the Ramsey-model utility function) and then comparing the correlations produced by repeated model simulations with the corresponding correlations in real-world macroeconomic data. More recent advancements in methodology have allowed some parameters of these models to be estimated in the simulation process.

RBC models recommend very different business-cycle policies than Keynesian models. Keynesian models emphasize the *inefficiencies* of cyclical fluctuations and especially the waste resulting from unemployed resources during recessions; RBC models claim that cyclical fluctuations may be *efficient* responses of the economy to unavoidable variations in the rate of technological progress. Thus, RBC advocates argue that government action to stabilize the economy through aggregate demand is inappropriate and useless.

As with any other theory, the major issue for the real-business-cycle model is whether it is capable of explaining the pattern of movements that characterize the modern business cycle. Opinions on the empirical performance of RBC models vary; these will be examined in detail in a later chapter.

B. Walrasian vs. Keynesian Explanations of Business Cycles

Why do we have multiple theories of business cycles?

Since basic forms of the currently popular theories have been around for at least thirty years, it might seem that by now the empirical evidence on business cycles ought to have pointed to one of the models as the most relevant. There are several reasons why it is difficult to achieve empirical consensus. First, recall that *all* models are extreme simplifications that are intended to illustrate a very limited set of macroeconomic patterns or phenomena. *Every* model has other phenomena that it cannot explain, and that detractors will use to discredit it. As a famous economist once said, "All models are wrong, but some are useful."

Another important reason is that the various models that we shall study during the remainder of the semester are, in many respects, ***observationally equivalent***. This means that similar outcomes are consistent with several theories, so observing these outcomes cannot be used to reject one theory in favor of the others. It does not mean, however, that the two theories are necessarily equivalent. Even theories that have very different implications for the optimal design of economic policy may share some of the same predictions about observable relationships among variables. In the absence of controlled experiments, we may find it very difficult to choose between theories based on slowly accumulating macroeconomic evidence.

A third reason for the multiplicity of models is that empirical evidence itself is subject to alternative methods of measurement and interpretation. An excellent example of this is the cyclical behavior of prices. Competing theories have different predictions for the correlation of price changes with output changes over the business cycle, so it might seem like one could easily choose between the theories by simply observing this correlation. But empirical studies can be found to support either procyclical or countercyclical price movements; the outcome depends on what time period and country is studied and on the particular method used to assess correlation. To take one among several measurement issues, it matters greatly whether one considers the cyclical behavior of the price level or the inflation rate.[2] There is enough disagreement in the evidence on this issue that proponents of both classes of models can claim that the cyclical behavior of prices supports their viewpoint. Similarly, authors using different

[2] See, for example, Lines 37 and 41 of Table 2 in Stock and Watson (1999), which show that cyclical movements in the level of the GDP price deflator are *negatively* correlated with output movements, while the inflation rate of the deflator is *positively* correlated with output.

methods and data sets have found real wages to be procyclical, countercyclical, and acyclical. Each of these possibilities is supported by one or more business-cycle models.

Another reason for disagreement is that some macroeconomists seem to have a near-religious belief in the validity of particular theories. Under these circumstances, evidence that would falsify a model in the eyes of most impartial scientists may be insufficient to dissuade a "true believer." Unfortunately, as noted just above, empirical evidence is rarely unanimous or conclusive, since proper measurement of variables and exact specification of economic relationships are nearly always open to question. Thus, the macroeconomic debate over the nature and causes of business cycles seems certain to continue for the foreseeable future.

Classification of business-cycle models

Early business-cycle research was dominated by theories of ***endogenous cycles***, in which the economy follows a cyclical trajectory even in the absence of external shocks. In these theories, a boom lays the seeds for its own demise and for the ensuing slump.[3] These endogenous-cycle models have fallen out of favor in recent decades and have largely been superseded by ***impulse-propagation models*** in which business cycles result from the response of the economy to exogenous shocks.[4] Our analysis will focus exclusively on the latter class of models.

There are many individual theories within the class of impulse-propagation models that vary in a number of ways. One basic taxonomy is between

- Theories that retain the ***Walrasian*** or ***neoclassical*** assumption that prices and wages are perfectly flexible and that supply equals demand in every market at all times, and
- ***Keynesian*** theories featuring markets that do not clear because of imperfect adjustment of wages and prices.

You may recall from basic microeconomic theory that the Walrasian equilibrium model is based historically on the work of Leon Walras in the 19th century. Walras advanced the idea that markets behave as though there is an auctioneer who calls out a range of possible prices and gets demand and supply information at each price. This ***Walrasian auctioneer*** then determines the price that would balance supply with demand and establishes the market price at this level before trading begins. All trading occurs at the equilibrium price in a Walrasian market.

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[3] Examples of endogenous-cycle models from the early literature include Goodwin (1948) and Hicks (1950).

[4] Much of this history is discussed in detail in Chatterjee (2000).

Of course, no one pretends that a Walrasian auctioneer exists in common real-world markets. However, actual markets may behave in a similar manner if any condition of excess supply or excess demand quickly leads to a price change that eliminates the disequilibrium. Prices are very flexible in some markets, so (for them) the Walrasian assumption is probably a reasonable one. For example, few would claim that prices on the New York Stock Exchange remain out of equilibrium for more than a minute or two—or, with modern computer-based trading algorithms, for more than a millisecond or two. At the other extreme, prices in labor and housing markets where products are highly differentiated and information is costly may be a great deal stickier and a condition of excess demand or supply may persist in a period we call the "short run."

Many macroeconomists believe that market clearing is a reasonable assumption over a medium-to-long time horizon, but that prices are likely to exhibit some stickiness in the short run. This has led to a collection of "new Keynesian" models that we shall study in Romer's Chapters 6 and 7, with imperfect price flexibility in the short run and Walrasian assumptions in the long run. However, even among those conceding that price rigidity is important disagreement arises (1) over how long, in terms of actual time, the relevant short run and long run are and (2) over how much importance should be attached to the short run vs. the longer run in deciding economic policy.

Macroeconomists divide broadly into two camps. ***Neoclassical*** macroeconomists, sometimes called by the 1950s-vintage name ***monetarists***, generally view Walrasian market clearing as an appropriate paradigm for analysis. They argue that prices are relatively flexible and/or that long-run considerations are more important than short-run considerations. On the other side of the debate, ***Keynesians*** emphasize rigidities and coordination failures that prevent markets from clearing. They often place greater importance on short-run outcomes than on long-run effects. We examine one major branch of the neoclassical view here then turn to the basic outlines of the Keynesian model in future chapters.

Within the neoclassical camp, there are two main kinds of models. Following the work of Robert Lucas, Thomas Sargent, Neil Wallace, and others in the early 1970s, the first collection of models was based on continuous market clearing in an environment where agents have *imperfect information*. Romer discusses these models in the last section of Chapter 6.

The other broad set of neoclassical models follows Nobel laureates Finn Kydland and Edward Prescott, who developed the ***real-business-cycle*** (RBC) model in the 1980s.[5] Unlike the imperfect-information models, the pure RBC model introduces no imperfections whatsoever to the system of perfect competition, perfect information, and instantaneous market clearing. The RBC model is basically a ***stochastic growth model***,

[5] Kydland and Prescott (1982) is usually cited as the seminal paper in the RBC literature.

extending our full-employment growth models to allow for random fluctuations in the rate of growth of productivity and natural output.

After the Great Depression, the conventional wisdom in macroeconomics was that the business cycle was a non-Walrasian phenomenon. Under Walrasian assumptions, the level of output is always at the natural level of output—the level that is consistent with full employment of labor and full utilization of capital, given the state of available technology. Walrasian models explain business cycles as fluctuations *in* the natural level of output, rather than fluctuations of actual production *around* the natural level. The traditional view was that technological progress and changes in the labor force and capital stock were usually smooth, trend-like movements like those modeled in economic growth theory. If the determinants of natural output move smoothly rather than cyclically then the Walrasian model cannot explain business cycles.

Two insights of the RBC modelers have enabled them to construct a Walrasian competitive equilibrium model with business cycles. First, they recognized that technological progress does not necessarily occur smoothly but may instead have ebbs and flows, perhaps even periods of regress. Second, RBC modelers devised endogenous propagation mechanisms that could cause changes in the rate of technological progress to affect other variables in a way that leads to co-movements that resemble those we observe in business cycles. The central question that RBC proponents' models have attempted to answer affirmatively is: Can observed realistic movements of variables over the business cycle be explained *without stepping outside the competitive Walrasian model?* They claim that the ability to replicate real-life co-movements among major macroeconomic variables using a model that is purely Walrasian means that Keynesian concepts of wage and price stickiness are not essential to explain economic fluctuations.

Despite the large amount of attention that RBC models received in the 1980s and 1990s, most macroeconomists remain skeptical. Since the Great Depression, the Keynesian view has dominated macroeconomics: recessions are viewed as reductions in output below the natural level, not declines in the natural level itself. Proponents of RBC models were fighting an uphill battle for acceptance of their ideas. Because of this, much of the literature on real business cycles carries a tone of persuasion that might seem unusual to readers (such as you) who learn about RBC models *before* learning about the much-longer-established Keynesian tradition of business-cycle analysis.

The intellectual combat between RBC modelers and Keynesians obscures a crucially important point: it is entirely plausible that business-cycle fluctuations reflect *both* uneven movements in natural output *and* movements in actual output away from the natural level. Even if RBC models are *capable* of reproducing business-cycle movements, this does not mean that the sources of fluctuations emphasized by these models are the *only* source of business cycles. Most macroeconomists agree that the oil shocks of the 1970s had a substantial macroeconomic impact, as predicted by RBC models and verified consistently by much macroeconomic research. However, most also believe that monetary and fiscal policy changes and other shifts in aggregate demand have a strong influence on short-run economic activity, as in Keynesian models.

C. Understanding Romer's Chapter 5

Romer's baseline model

Many of the components of the model that Romer lays out in section 5.3 should be quite familiar, although Romer sets the RBC model in discrete rather than continuous time.[6] The Cobb-Douglas production function in (5.1) should be very familiar by now. The capital-stock adjustment equation (5.2) is a standard discrete-time formulation that includes government spending. However, this equation slips in an assumption that turns out to be important later on. In equation (5.2), investment that takes place in period t does not become productive until period $t + 1$. To see this, note that an increase in I_t affects K_{t+1} and therefore Y_{t+1}, not K_t and Y_t.[7]

Equation (5.3) equates the real wage w to the marginal product of labor and (5.4) does the same for the interest rate and the marginal product of capital. We will study the investment decision in greater detail later in the course and at that time the analytical basis of (5.4) will be made clearer. For now, we develop an intuitive argument for why this equation represents the optimal amount of capital to be used in production.

Labor input is always "rented" by the firm—it cannot own workers—so the labor market is a market for labor services. However, in the real world most firms own a

[6] Many of the models developed in the research literature are continuous-time; the conclusions from these models are similar to Romer's. The analysis of continuous-time random processes is difficult to explain intuitively, so discrete time is an easier choice for textbook exposition. Moreover, the implementation of models in numerical simulations must also be done with discrete time intervals, so even models developed in continuous time must be estimated and simulated in discrete-time form.

[7] This equation could alternatively be specified as $K_t = K_{t-1} + I_t - \delta K_{t-1}$, which would make period t investment part of the productive capital stock in period t. Since investment presumably occurs throughout the period, neither assumption seems strictly correct. If our period is a year, January investment may be used in production throughout the current year, but December investment is not. One of the disadvantages of using discrete-time analysis is that one must make arbitrary decisions of this kind. Romer's choice is probably the better one, since much of investment is in large projects (such as a new factory) that require many months of expenditures before entering the productive capital stock. Some capital projects may even require investment over two or more years before they can be used.

large share of the capital goods whose services they use in production. Owning capital goods ties up some of the firm's financial resources, which could otherwise be used in other ways. For example, instead of owning a dollar's worth of physical capital, the firm could own a one-dollar bond that would pay r_t per year in (real) interest.

For the firm to be content to own and use a marginal unit of capital instead of trading it for an equivalent amount of bonds, the capital must earn a rate of return equal to the interest on the bond. The right-hand side of Romer's equation (5.4) is the marginal product of capital less depreciation, which measures the net rate of return on the marginal unit of capital. Thus, equation (5.4) expresses the equilibrium condition for the amount of capital owned by the firm as the equality between marginal returns on two alternative uses of resources: the rate of return on a bond (the interest rate) and the marginal rate of return on capital.

The utility function in (5.5) differs in one important way from the ones we used in our growth models: it includes leisure as well as consumption. In growth theory, we suppressed the role of leisure in the utility function by assuming that working time was fixed independently of the consumption choice. We could defend that assumption by noting that long-run changes in per-capita labor supply (hours worked) seem to occur slowly and smoothly and can thus be captured by setting n to be the growth rate of hours worked rather than just the population growth rate. In contrast to growth theory, movements in employment (and unemployment) are central to the analysis of business cycles. If these movements are to be explained in a Walrasian framework where demand equals supply, there must be fluctuations in the quantity of labor supplied. To model labor supply, we take account of the labor-leisure tradeoff and so we enter leisure into the utility function. Leisure and consumption enter additively in (5.7), which implies that their marginal utilities are independent of one another (*i.e.*, the second cross partial derivatives are zero). This assumption makes the analysis *much* easier.

The logarithmic form of the utility function in (5.7) deserves special attention as well. We encountered the combination of logarithmic utility function and Cobb-Douglas production function in section 2.10, where Romer uses it to simplify the dynamics of the Diamond overlapping-generations growth model. Log utility is the special case of the CRRA utility function where $\theta = 1$. This case has a very useful property: saving is independent of interest rates because the wealth (income) and substitution effects of a change in r exactly offset one another.

Can we defend the use of this special utility function on empirical grounds or is it merely a way of simplifying the analysis? It is very difficult (impossible?) to estimate underlying utility function parameters such as θ from economic data. Some evidence in support of $\theta = 1$ comes from the apparent absence of large effects of interest rates on saving. Ultimately, the relevance of the model and its parts must be tested by comparison with actual data. If the model is able to mimic the data satisfactorily, then perhaps its assumptions (including log utility) are roughly congruent to actual behavior. If not, then we must consider log utility alongside the model's other assumptions as possible culprits.

Introducing random shocks

The biggest methodological difference between the RBC model and the growth models of Romer's chapters 1 through 4 is that the RBC model is ***stochastic***, meaning that it includes random elements. Without these random shocks the Ramsey model (and the RBC model based on it) converges to a balanced-growth path without cycles; it is the shocks that introduce cyclical behavior in the model.

Two random variables are introduced: shocks to the growth of technology and shocks to government spending. The random disturbances enter Romer's model in a rather complicated way. There are three "versions" of random variables for A and for G. The first version is the economic variable itself: A and G. The log of each of these variables is determined as the sum of a ***trend*** component and a ***deviation from trend***. The deviation from trend is the second version of the random variable, which is then decomposed into two parts, one representing the tendency for past deviations to persist and the other measuring the new ***shock*** to the deviation, which is the lowest-level random variable.

The trend level of $\ln A$ is $\bar{A} + gt$, where $\bar{A}$ is a constant and gt reflects the assumption that the level of productivity fluctuates around a trend that is growing at rate g. (This is an example of a "trend-stationary" stochastic process.) Government spending is assumed to vary around a trend growth rate of $g + n$. That is the same as the growth rate of real output, so there is no long-run tendency for the share of output devoted to government spending to rise or fall. The trend level of $\ln G$ is $\bar{G} + (g+n)t$.

At any point in time, both productivity and government spending are subject to deviations above or below their trend levels. The deviation in A is $\tilde{A}$ and the deviation in G is $\tilde{G}$. Thus, combining the trend levels with the deviations, we get Romer's equations (5.8) and (5.10), which are reproduced below:

$$\ln A_t = \bar{A} + gt + \tilde{A}_t,$$
$$\ln G_t = \bar{G} + (n+g)t + \tilde{G}_t.$$

Finally, we model the deviations themselves. The easiest way to model the deviations would be as random disturbances that are totally independent over time. But if each period's deviation from trend was independent of the ones before and after, then we would expect to see productivity and government spending chaotically jumping back and forth above and below their trend lines. This is not very realistic; evidence suggests that both productivity and government spending tend to have sustained periods in which that are above (or below) their trends. We therefore choose to model the deviations as having some degree of ***persistence***.

The deviations from trend $\tilde{A}$ and $\tilde{G}$ are assumed to follow ***first-order autoregressive processes*** as described by (5.9) and (5.11).[8] The first term in these equations, $\rho_A \tilde{A}_{t-1}$ and $\rho_G \tilde{G}_{t-1}$, allows the previous period's value of the deviation to carry over (positively or negatively) into the current period. If ρ_A and ρ_G are positive, which we always assume, then a positive (negative) deviation from trend would tend on average to be followed by a succession of positive (negative) deviations, returning gradually to zero.

The ***shocks*** ε_A and ε_G are ***white-noise*** random processes, meaning that they cannot be predicted ahead of time. Thus, when consumers form their consumption plans, they will use the expected value (zero) of the future shocks ε_A and ε_G rather than the unknown actual values.

If an important goal of a business-cycle model is to explain the persistence of macroeconomic fluctuations, then starting out with persistent exogenous deviations could be viewed as cheating. It *assumes exogenous persistence* rather than *explaining persistence endogenously*. This is a major criticism of models such as the one Romer presents. By analyzing the model when ρ_A and ρ_G are zero and then when they are positive, we can determine how much of the persistence in business-cycle fluctuations is explained endogenously by the model and how much is attributed to the persistence of shocks hitting the economy.[9]

Analysis of household behavior

Romer examines the household maximization problem gradually by starting with one period, then moving to two periods, and eventually to more than two. People in this model make simultaneous substitution choices over two basic dimensions: present vs. future and leisure vs. consumption. In principle, even in a two-period model there is substitution between all six pair-wise combinations of these dimensions: present con-

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[8] Time-series processes of this kind are called "autoregressive" because the variable is a function of its own past values. "First-order" refers to the fact that only one lagged value appears in (5.9) and (5.11).

[9] The two papers that originated the literature on real business cycles had more elaborate propagation mechanisms than those in Romer's version or in much of the subsequent literature. Kydland and Prescott (1982) assumed that investment projects require several periods to complete ("time to build"), while Long and Plosser (1983) used an input-output structure where changes in demand take multiple periods to work their way through the purchasing of inputs to production.

sumption vs. future consumption, present leisure vs. future leisure, present consumption vs. present leisure, future consumption vs. future leisure, present consumption vs. future leisure, and present leisure vs. future consumption. And in a model with more than two periods, there are multiple "futures."

We generally focus on just the first three of these. We will analyze three sets of first-order conditions to represent the equilibrium conditions corresponding to these three tradeoffs. The fourth tradeoff (future consumption vs. future leisure) is just the same as the third in that both relate to the contemporaneous choice between consumption and leisure. The appropriate first-order conditions for the fifth and sixth tradeoffs can be shown to be redundant with our separable utility function: they are satisfied if the first four conditions are satisfied.

The two-period model is sufficient to illustrate one of the most important features of RBC models: ***intertemporal labor substitution***. Put very simply, the RBC model asserts that people will tend to allocate their labor effort over time so that they work more when the (real) wage rate is high and partake in more leisure at times when the wage is low. The same substitution mechanism also implies that high interest rates should motivate households to work more now since today's wage is worth more relative to the present value of the future wage. Romer's equation (5.21) shows formally how labor effort in the two periods is related to relative wage rates in the periods and the real interest rate.

The substitution between present and future consumption in the RBC model is generally quite similar to that of the Ramsey and Diamond models. The analysis is in discrete time, like the Diamond model, but Romer chooses continuously compounded discounting as in the Ramsey model. The main new feature is the presence of uncertainty about future variables due to the introduction of the random shocks. Since future values of shocks are unknown, any future-dated variables must be replaced by their expectations as of the present, which are denoted by the expectations operator $E_t[\cdot]$.

Although the analysis is made more complicated by the presence of the expected value operator in the expression, you should recognize a variant of our old friend the ***intertemporal consumption-equilibrium*** (Euler) equation in equation (5.23). Once again, as in equation (2.20) in the Ramsey model and equation (2.48) in the Diamond model, households want a rising or falling consumption path depending on whether $r > \rho$ or $r < \rho$.[10]

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[10]This condition is only approximate in equation (5.23) because Romer uses the exponential (continuously compounded) form of discounting for utility, but the discrete-time $1/(1+r)$ form of interest accumulation. Thus, we end up with a mixture of exponentials and quotients. Note that for small values of ρ, $e^{-\rho} \approx 1/(1 + \rho)$, so once again the relationship (in expectation) between current and future consumption depends on something very like $(1 + r)/(1 + \rho)$.

A new dimension of household equilibrium in this model is the (intratemporal) equilibrium between consumption and leisure. Since leisure and consumption are analogous to any pair of utility-yielding commodities, the ratio of their marginal utilities should match the ratio of their prices at the point of utility-maximization. (Geometrically, this is the condition for the indifference curve to be tangent to the budget line.) The relative price in this case is the real wage w, with the ratio of marginal utilities contributing the other side of equation (5.26).

Each of these three conditions will hold in every period or pair of periods: intertemporal labor supply decisions will satisfy a variant of equation (5.21) for every pair of periods, intertemporal consumption satisfies (5.23), and the contemporaneous tradeoff between consumption and leisure (labor) at each date must follow (5.26).

Solving the model?

As Romer points out, this model is too complicated to be solved analytically. We can, however, establish intuitively or analytically some of the properties of a solution to the model. The model itself consists of the production function (5.1), the capital accumulation equation (5.2), the wage and interest rate marginal-product conditions (5.3) and (5.4), the population-growth equation (5.6), the specifications of the stochastic evolution of technology and government spending (5.8) through (5.11), the three sets of household-equilibrium conditions that are discussed in the previous paragraph, and the identities $c = C/N$ and $l = L/N$ on page 197. Many of these equations are dynamic in nature: the capital-evolution equation, the equations of motion for A, G, and N, and the two sets of intertemporal-substitution conditions. The others hold at every moment of time. The endogenous variables whose paths are to be determined by these equations are Y, N, A, G, C, c, L, l, w, r, and K.[11]

In principle, we could try to reduce these equations to a smaller set through substitution and use a phase diagram to try to explain the dynamics of the system. However, one cannot reduce the system easily to two variables to fit on a plane, so this method does not simplify the analysis enough to be illuminating.

On a more fundamental level, what would a solution to this model look like? The solution of a model expresses each of the endogenous variables as a function of only exogenous and lagged variables. (This is sometimes called the ***reduced form*** of the model.) If all of the variables listed above are endogenous, what is left to be exogenous? Other than the time trend (which affects A and G), the only exogenous "driving" variables in the system are the random shock terms ε_A and ε_G. Thus, a complete solution

[11]Note that because the model is dynamic you cannot just count variables and equations. The household behavioral conditions are not independent of one another: if the condition for balance of c_t and c_{t+1} holds and the conditions for balance of l and c hold at both t and $t + 1$, then the resulting relationship between l_t and l_{t+1} will surely satisfy its condition as well.

to the model would express the time path of each of the endogenous variables as a function of the time paths of these shocks and of the time trend. Romer does not achieve or approximate such a solution in Chapter 5, but you will have the opportunity to solve the model through numerical simulation in a homework project.

What we would learn from analyzing the solution (if we could find one) is the nature of the response of each variable to a one-unit shock to technology or to government spending. Because of the complexity of the models, researchers usually perform these experiments through numerical simulation rather than by algebraic solution. First, the model is "calibrated" by assuming values for unknown parameters such as ρ in the utility function and α in the production function. Next, random values are generated for the shock innovations (ε) according to probability distributions that they have been assumed to follow. Then a computer program calculates the values of the variables of the model in each period given the shocks and adjusts prices (wages, interest rates, etc.) until all markets are in balance.[12] By repeating this simulation process many times for different sets of randomly generated disturbances, we can characterize the basic properties of the model: the correlations among variables at various leads and lags, relative variances of variables, and patterns of autocorrelation and persistence of individual variables. If these properties of the simulated model look like business cycles, then the model is deemed to be capable of representing business-cycle behavior.[13]

Romer's "special case"

In section 5.5, Romer attempts an approximate solution to a special case of the model, where there is no government and where the depreciation rate is 100%. As Romer notes, these are not realistic conditions, but rather must be defended on the basis of convenience. He uses this simplified model as a base case then considers how the more general model might compare to it.

The principal effect of the simplification is that the saving rate is constant. His derivation of the saving rate in (5.33) is quite straightforward except, perhaps, for how he deals with expectations. In light of the discussion on page 200 of how one cannot separate out the expectation of a product (or quotient) of two random variables as the product (quotient) of their expectations, it may seem like Romer is cheating in equation (5.30) when he brings α, N_{t+1}, s_t, and Y_t outside the expectation operator. However, none of these is random from the standpoint of the agent forming expectations.

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[12] Note that the dynamics of this simulation are complex. Current decisions depend on expectations of future values of the variables; future values of the variables depend on current decisions. One cannot in general simply step through time recursively to solve these models but must solve all periods of the simulation "at once."

[13] Judd (1998) provides an excellent introduction to the numerical analysis methods used to solve dynamic economic models such as RBC models.

Because the population increases in a totally predictable way, N_{t+1} is known with certainty. The other variables are known (or being determined by the consumer) at date t and the α parameter is assumed known. Thus, filling in a step that Romer omits,

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\begin{aligned}
-\rho + \ln E_t\left[\frac{\alpha N_{t+1}}{s_t\left(1-s_{t+1}\right)Y_t}\right] &= -\rho + \ln\left[\frac{\alpha N_{t+1}}{s_t Y_t} E_t\left(\frac{1}{1-s_{t+1}}\right)\right] \\
&= -\rho + \ln\alpha + \left(\ln N_t + n\right) - \ln s_t - \ln Y_t + \ln E_t\left(\frac{1}{1-s_{t+1}}\right).
\end{aligned}
$$

The next step is to note that if we have a constant saving rate, as Romer assumes in this section, then $s_{t+1} = s_t$, which is known. Thus, we can eliminate the last expectation from the right-hand side of (5.31) and derive the simple expression (5.33) for the saving rate.

The remaining equations on pages 203 and 204 analyze the labor-supply decision by plugging in the Cobb-Douglas formula for the marginal product of labor (the equation in text between (5.34) and (5.35)) and solving. The result is that under the simplifications we have introduced, all of the substitution and income effects of changes in wages and interest rates associated with a productivity shock exactly balance, leaving labor supply unaffected by A. Although this is a remarkable and seemingly counterintuitive result, aggregate labor supply does tend to be remarkably stable through various shocks to the economy.[14]

In the next section, Romer describes the dynamic behavior of real output in this simplified model. As you might expect, assuming 100% depreciation eliminates a lot of the complicated dynamics of the model. If capital lives many periods, then high investment during a boom will affect productive capacity for many ensuing years. By assuming complete depreciation, that cannot happen here. However, the model still has some degree of endogenous persistence due to the one-period lag in installation of capital.[15]

---

[14]This assumption is particularly problematic. As discussed in Mankiw (1989) and elsewhere in the literature, one of the major problems for RBC models is explaining why labor input is so sensitive to changes in A if the labor market clears continuously. If all workers are on their labor-supply curves, then something must be inducing them to work a lot more in booms and a lot less in slumps. Presumably the wage, in response to productivity fluctuations, is that something, but if we assume that labor supply is insensitive to wage changes, then we cripple that mechanism.

[15]See the discussion in the previous section about the choice of whether current-period investment becomes part of the capital stock in the current period or in the next period.

In equation (5.40), Romer defines the new variable $\tilde{Y}$ to be the deviation of log output from its "no-shock" value, which is the value of Y that would occur if all shocks were zero. We can think of this as decomposing Y into trend and cyclical components. Specifically, if we define $\bar{Y}_t$ to be the no-shock or trend level of the log of output in period t, then $\tilde{Y}_t \equiv \ln Y_t - \bar{Y}_t$ is the cyclical component. Working with Romer's equation (5.39), if all shocks (in all time periods) were zero, then $\tilde{A}_t = 0$ and $\ln Y_{t-1} = \bar{Y}_{t-1}$. Thus, from (5.39),

$$\bar{Y}_t = \alpha \ln \hat{s} + \alpha \bar{Y}_{t-1} + (1-\alpha)\left(\bar{A} + gt\right) + (1-\alpha)\left(\ln \hat{\ell} + \bar{N} + nt\right),$$

and

$$\begin{aligned}\tilde{Y}_t \equiv \ln Y_t - \bar{Y}_t &= \alpha\left(\ln Y_{t-1} - \bar{Y}_{t-1}\right) + (1-\alpha)\tilde{A}_t \\ &= \alpha \tilde{Y}_{t-1} + (1-\alpha)\tilde{A}_t.\end{aligned}$$

This is Romer's equation (5.40).

Romer derives the dynamic equation for cyclical output in (5.42) through some simple, but tricky, manipulation. This equation shows that cyclical output is affected by the current innovation to the productivity shock $\varepsilon_{A,t}$ and by two lags of its own values. One of these lags reflects the first-order autoregressive process that was assumed for the productivity deviations. Notice that the second lag disappears if ρ_A is zero. The other lag results from the assumption that investment in the current period forms the capital stock of the following period.

Method of undetermined coefficients

Suppose that we have a static linear (or linearized) model that we would like to solve for a set of N endogenous variables $y_1, y_2, \ldots y_N$. We have N linear equations each of which involves a subset of these endogenous variables and a set of (lagged and) exogenous variables x_j. Linear algebra teaches us that under appropriate conditions of independence and consistency of the equations, there is a unique solution that expresses each y_i as a linear function of the complete set of x_j variables. (Again, what makes this a *solution* is that each endogenous variable is expressed *solely* as a function of exogenous variables.)

There are various methods for solving such systems of linear equations; we can solve it by successive substitution or Cramer's Rule, for example. But while we are assured that there *is* a unique linear solution, these methods can yield complicated expressions that are very difficult to interpret. An alternative method that sometimes leads to a more intuitively logical process of solution is the ***method of undetermined***

coefficients. Under this method, one posits the form of the solution, then essentially solves both backward and forward to relate the coefficients of the solution (reduced form) to those of the structural equations.

In the case of the model of section 5.6, Romer posits a solution for the log-linearized system of household behavioral equations in the form of equations (5.43) and (5.44). These equations express the log-deviations (or cyclical components, if you prefer to think of them that way) of the two household choice variables as linear functions of the log-deviations (cyclical components) of the three state variables. To see how Romer uses this method to get information about the solution, let us examine equations (5.46) and (5.47). He derives (5.46) by using the same Taylor-series approximation technique that he used in Chapter 1 to examine the speed of convergence to the Solow steady state (see Chapter 2 of the coursebook). He moves from equation (5.46) to (5.47) by substituting backward from the posited solutions (5.43) and (5.44) into (5.46).

He then sets each variable's coefficients equal on both sides of the equation to get (5.48) through (5.50). For example, if you collect terms involving $\tilde{K}$ on the left-hand side of (5.47), the result will be the expression on the left-hand side of (5.48). This is set equal to the $\tilde{K}$ coefficient from the right-hand side of (5.47). Performing the same procedure for $\tilde{A}$ and $\tilde{G}$ yields equations (5.49) and (5.50), respectively.

Why must we insist that *each* pair of coefficients be equal? After all, there are some combinations of $\tilde{K}$, $\tilde{A}$, and $\tilde{G}$ for which equation (5.47) will hold even if each set of coefficients does not match up. However, equation (5.47) must hold for *all* sets of values of $\tilde{K}$, $\tilde{A}$, and $\tilde{G}$. In particular, it must hold when $\tilde{G} = 0$ and $\tilde{A} = 0$, in which case equation (5.47) reduces to only the terms on the two sides involving $\tilde{K}$. If we divide both sides of this reduced equation by $\tilde{K}$, we get exactly equation (5.48). Similarly, since (5.47) must hold in the case that $\tilde{K} = 0$ and $\tilde{G} = 0$, we can reduce it to the terms involving only $\tilde{A}$, then divide by $\tilde{A}$ to get equation (5.49).

While equations (5.48) through (5.50) are not full solutions of the household-choice model, they allow us to characterize certain properties of the solutions, as Romer does in the three following paragraphs. In the section on the intertemporal first-order condition (beginning on page 210), Romer extends the analysis to a posited solution for the dynamic evolution of $\tilde{K}$ in equation (5.53). He then shows that it is possible to substitute this further and characterize the dynamic behavior of the consumption model, though he asserts correctly that little would be gained by anyone (other than ink-producers) from including the resulting complicated expressions in the text.

Instead, Romer presents a series of graphs showing the effects on the model's major endogenous variables of one-time innovations to the productivity and government spending shocks. These graphs were obtained by numerical, rather than algebraic, solution of the model using the parameter values he describes on page 211. These graphs are highly representative of the properties of simple RBC models, so you should devote sufficient attention to them to assure that you understand them.

D. Simulating Romer's RBC Model

A large share of current research in macroeconomics uses simulation of dynamic, stochastic general-equilibrium models. The model of Romer's Chapter 5 is a useful subject for such simulation, a task that you will be asked to undertake in an upcoming homework project. This section of the chapter provides some background preparation for the tasks that you will do in that project.

Some small changes in notation

Romer's model is a variation of the model discussed by Campbell (1994). You may wish to refer to this paper when doing the assignment. Romer's version makes two essential changes to the Campbell model: (1) Romer adds government spending as a component of output demand and (2) Romer's model restricts the labor/leisure term of the additive utility function to be logarithmic rather than using the CRRA form.

Our focus here is on business cycles, so we approximate the *deviations* of the variables around the steady state and ignore changes in the steady-state values themselves over time. In Romer's model, the population N grows at a constant rate n and never deviates from its steady-state path. Thus, the deviations from steady-state never arise from changes in population and we can suppress population growth by setting $n = 0$. Without loss of generality, we set the constant value of the population at $N = 1$.

In converting the model to a log-linear approximation to deviations from the steady state, it is useful to follow Campbell's notation convention of using lower-case letters to represent the logs of capital-letter variables. This requires some minor notation changes from Romer's Chapter 5. For example, $k \equiv \ln K$, not K/AL as in the growth models that we have studied. Moreover, in this assignment, $c \equiv \ln C$ and $l \equiv \ln L$, whereas Romer's Chapter 5 uses c and l to refer to C/N and L/N, respectively. (We do not need the per-capita variables here because we have set $N = 1$.) We will try to be careful in the following pages to remind ourselves of the exact definitions of these variables.

In addition, we shall make a couple of trivial modifications to Romer's equations.

- We use K_t to refer to the capital stock at the *end* of period t rather than (as Romer uses it) the *beginning*. This is a mere change in timing convention that conforms better to the requirements of the Dynare software that we use for simulations; it does not change the structure of the model at all.

- We define R_t to be the "gross" return on capital, which is one plus the net return that Romer calls r_t. The gross return on a bond includes the repayment of the bond plus the interest earned. Following our logarithmic-notation convention, we will define r_t to be $r_t \equiv \ln R_t \cong R_t - 1$, so r_t is (within our approximation) the same as Romer's variable.
- To reserve lower-case letters for logs, we denote the real wage by W and its log by w.
- Finally, because we will need g to be the log of G, we will denote the growth rate of productivity (Romer's g) as γ, with $e^{\gamma} \cong 1 + \gamma$.

The Romer model in our notation

We now proceed to state the equations of the Romer model with these notation changes. To make it convenient to refer back to the full set of equations in a compact location, all eight equations are listed first, with descriptions following:

$$Y_t = K_{t-1}^{\alpha}\left(A_t L_t\right)^{1-\alpha} \tag{1}$$

$$K_t = \left(1-\delta\right)K_{t-1} + Y_t - C_t - G_t \tag{2}$$

$$W_t = \left(1-\alpha\right)\left(\frac{K_{t-1}}{A_t L_t}\right)^{\alpha} A_t, \tag{3}$$

$$R_t = \alpha\left(\frac{A_t L_t}{K_{t-1}}\right)^{1-\alpha} + \left(1-\delta\right). \tag{4}$$

$$\frac{1}{C_t} = e^{-\rho} E_t\left(\frac{R_{t+1}}{C_{t+1}}\right). \tag{5}$$

$$bC_t = \left(1-L_t\right)W_t. \tag{6}$$

$$\begin{aligned} \ln A_t &\equiv a_t = \gamma t + \tilde{a}_t, \\ \tilde{a}_t &= \rho_A \tilde{a}_{t-1} + \varepsilon_{A,t}, \end{aligned} \tag{7}$$

$$\begin{aligned} \ln G_t &\equiv g_t = \overline{g} + \gamma t + \tilde{g}_t, \\ \tilde{g}_t &= \rho_G \tilde{g}_{t-1} + \varepsilon_{G,t}. \end{aligned} \tag{8}$$

Equation (1) is Romer's (5.1), the production function. The only change here is that the beginning-of-period capital stock is now called K_{t-1} rather than K_t.

Equation (2) is Romer's (5.2), the capital-accumulation equation. Again, the only change is the adjustment of the time subscripts of capital.

Equation (3) is Romer's (5.3), which sets the real wage equal to the marginal product of labor. The sole change here is that we are using capital W for the real wage (and reserving lower-case w for its log).

Equation (4) is Romer's (5.4), which sets the interest rate equal to the marginal product of capital minus the depreciation rate. Note that we have added one to both sides of (5.4) to get (4): R_t on the left is $e^{r_t} \cong 1 + r_t$, and the term $-\delta$ on the end of the right-hand expression has changed to $1 - \delta$.

Equation (5) is Romer's (5.23), the Euler equation for intertemporal consumption. We use capital C here because our variables are already in per-capita terms (and we want lower-case c to be its log) and our $R_{t+1} \equiv 1 + r_{t+1}$.

Equation (6) is Romer's (5.26), which equates the marginal utility of consumption to that of leisure. Again, C and W are in capitals rather than Romer's lower-case, but otherwise the equations are identical.

Equations (7) and (8) are the equations of motion for technology and government spending, corresponding to Romer's (5.8) through (5.11). The equations are identical except for introducing lower-case a and g as the logs of A and G, setting $n = 0$ (because we ignore population growth), and replacing Romer's growth rate g by γ.

Equations (1) through (8) constitute eight equations in Y, K, L, A, W, R, C, and G, plus lags of K and expected future values of C and R. To gain some insight into how such a model is solved, suppose that we consider an economy that starts at time zero in a steady state with a given, known value of K_0. Whatever shocks hit the system at time 0, we expect that the economy will re-converge to a steady state after some large number of periods, say T. We can iterate the evolution of capital forward through the T periods from its initial value K_0 using equation (2) (and the other equations of the model). Similarly, because we know that the shock will have died out by period $T + 1$, we assume that C and R will be back to their steady-state values by then. This allows us to solve the model as $8 \times T$ equations in the $8 \times T$ variables corresponding to periods 1 through T. This solution is much more feasible if the equations of the model are linear (not to mention making handing the expectations feasible), so our next step will be the express the model as a linear approximation of deviations from the steady state.

Calibration of the model

Recent advances in numerical simulation methods allow one to estimate parameters of the model based on macroeconomic data. (Indeed, Dynare is capable of doing such estimation.) However, the econometrics of this are beyond our scope and the emphasis here is less on achieving an ideal fit than on understanding how the model works, so we will assume a set of parameter values for the model.

In the first paragraph of Romer's Section 5.7, he suggests the following values for the parameters of a model (assuming that each period is one quarter):

- $\alpha = 1/3$, which corresponds roughly to capital's share of income,
- $\gamma = 0.005$, which is a 2% per year growth rate of productivity,
- $\delta = 0.025$, a 10% per year rate of depreciation,
- $\rho_A = \rho_G = 0.95$, which means that both productivity and government spending return to their steady-state paths at a rate of 5% per quarter.

The other parameters (ρ, b, and $\bar{g}$) will be set in such a way that the steady-state government/output ratio is $(G/Y)^* = 0.2$, the steady-state interest rate is $r^* = 0.015$ (per quarter), and steady-state $L^* = 1/3$ (meaning that our representative agent works one-third of the time).

Some key steady-state values

When we do the log-linear approximations to the model, it will be very useful to know some key values in the steady state. The long-run behavior of this model is essentially identical to that of the Ramsey growth model: compare equations (1), (2), and (5) to the corresponding equations from Chapter 2. Thus, we can confidently expect that Y, K, C, G, W, and A should all grow at constant rate γ on their steady-state growth paths and that R and L should be constant in the steady state. Indeed, we assumed in the calibration discussion above that the steady-state values are $R^* = 1 + r^* = 1.015$ and $L^* = 1/3$.

Because many of the variables are growing in the steady state, it makes sense to think about ratios among them that might be constant, much like the K/AL variable we used in the Solow and Ramsey growth models. It will be convenient later to know the steady-state values of three ratios: A_t/K_{t-1}, Y_t/K_{t-1}, and C_t/Y_t, which we denote respectively as $\left(\frac{A}{K_{-1}}\right)^*$, $\left(\frac{Y}{K_{-1}}\right)^*$, and $\left(\frac{C}{Y}\right)^*$.

We can immediately determine the steady-state value of the interest rate in terms of the model's parameters from (5). In the steady state, consumption is growing at rate γ so $C_{t+1} = e^{\gamma} C_t$. This means that

$$\begin{aligned} \frac{1}{C_t} &= e^{-\rho} \frac{R^*}{e^{\gamma} C_t}, \\ R^* &= e^{\rho+\gamma}. \end{aligned} \tag{9}$$

Given our calibrations of steady-state r and γ, this allows us to determine ρ. (For the values suggested above, $R^* \cong 1.015 = e^{\rho + 0.005}$, or $\rho = 0.0099 \cong 0.01$.)

From equation (4), in the steady state,

$$R^* = e^{\rho+\gamma} = \alpha\left[\left(\frac{A}{K_{-1}}\right)^* L^*\right]^{1-\alpha} + (1-\delta),$$

$$\left[\left(\frac{A}{K_{-1}}\right)^* L^*\right]^{1-\alpha} = \frac{e^{\rho+\gamma} - (1-\delta)}{\alpha},$$

$$\left(\frac{A}{K_{-1}}\right)^* L^* = \left[\frac{e^{\rho+\gamma} - (1-\delta)}{\alpha}\right]^{\frac{1}{1-\alpha}},$$

or

$$\left(\frac{A}{K_{-1}}\right)^* = \frac{1}{L^*}\left[\frac{e^{\rho+\gamma} - (1-\delta)}{\alpha}\right]^{\frac{1}{1-\alpha}} \cong \frac{1}{L^*}\left(\frac{\rho+\gamma+\delta}{\alpha}\right)^{\frac{1}{1-\alpha}}. \qquad (10)$$

Recall that we are setting $L^* = 1/3$ as a calibration parameter, so equation (10) gives us a steady-state value for the ratio of A to last period's K that depends only on parameters we assume that we know.

Following a similar process, you can show that the steady-state value of Y_t/K_{t-1} is

$$\left(\frac{Y}{K}\right)^* = \frac{\rho+\gamma+\delta}{\alpha}$$

and the steady-state value of C_t/Y_t is

$$\left(\frac{C}{Y}\right)^* = 1 - \left(\frac{G}{Y}\right)^* - \frac{\alpha(\gamma+\delta)}{\rho+\gamma+\delta}.$$

(Deriving these results is part of the project exercise.)

Log-linearization of deviations from the steady state

To log-linearize the model, we introduce an additional bit of notation. For a variable X_t, the log-deviation is defined as $\tilde{x}_t \equiv \ln X_t - \ln X_t^* = x_t - x_t^*$, where X_t^* is the value that X would have if the economy were in a steady state at t. The goal of log-linearization is the express the equations of the model as a linear function of these log-deviation variables. Some equations are already linear in the logs of the variables. Others are not linear in logs and must the approximated using Taylor series.

We will work through equations (1), (2), and (5) here. Equations (7) and (8) are already in terms of log-deviations. Part of your task for the project is to develop log-linear versions of equations (3), (4), and (6).

Equation (1) is an example of an equation that is already linear in logs. Such equations do not require approximation and are easy to handle. Taking the logs of both sides of (1) yields

$$y_t = \alpha k_{t-1} + (1-\alpha)(a_t + l_t), \tag{11}$$

in terms of the lower-case logged variables. Because this equation must hold in the steady state, we know that

$$y_t^* = \alpha k_{t-1}^* + (1-\alpha)(a_t^* + l_t^*). \tag{12}$$

Subtracting (12) from (11) gives us

$$\tilde{y}_t = \alpha \tilde{k}_{t-1} + (1-\alpha)(\tilde{a}_t + \tilde{l}_t). \tag{13}$$

Equation (13) is linear in the log-deviation (tilde) variables, so it is the first equation of our log-linearized system.

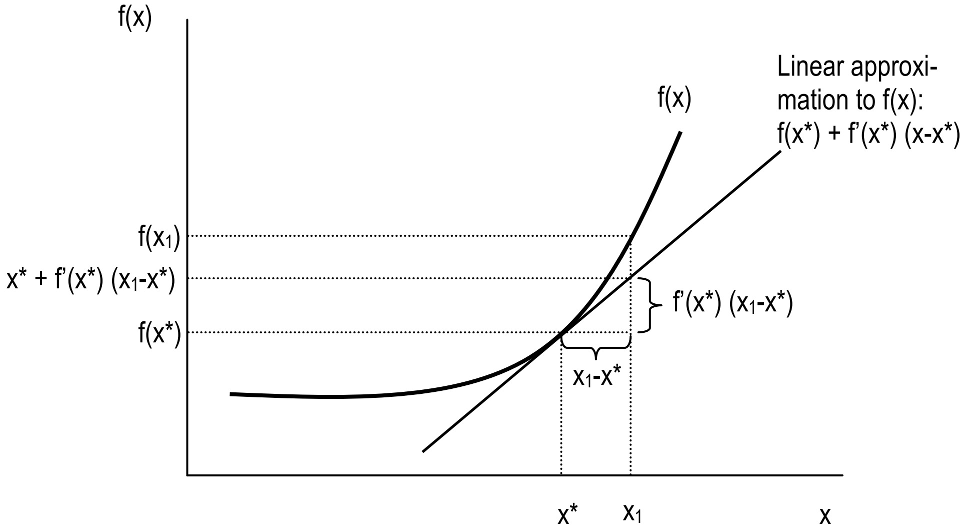
Equation (2) is linear in the *levels* of the variables, not in their *logs*. Thus, while it appears to be a very simple equation it is actually the most challenging to put in log-linear form, requiring us to use a Taylor-series approximation around the steady-state value.

Taylor's Theorem allows us to approximate a function by a straight line around a point at which the function's value is known. We used first-order Taylor series in our analysis of the rate of convergence in the Solow model. (See the related section near the end of Coursebook Chapter 2.) The basic idea of the Taylor series is that a general function $f(x)$ can be approximated near a specified value x^* by

$$f(x) \cong f(x^*) + f'(x^*)(x - x^*), \tag{14}$$

where, at the specified value x^*, the value of the function $f(x^*)$ and its first derivative $f'(x^*)$ are known or can be readily calculated.

In the figure below, the straight line is the linear approximation to the nonlinear function $f(x)$ around the value x^*. You can see that the first-order (linear) Taylor approximation is just the line that is tangent to the function at x^*. This approximation will be a good one if x is very close to x^* or if $f(x)$ is nearly linear.



The Taylor approximation of equation (2) will be accomplished by transforming the equation into two additive parts, then taking the approximation of each part. We begin by dividing both sides of (2) by K_{t-1} to get

$$\frac{K_t}{K_{t-1}} = (1-\delta) + \frac{Y_t}{K_{t-1}} - \frac{C_t}{K_{t-1}} - \frac{G_t}{K_{t-1}}$$
$$\frac{K_t}{K_{t-1}} - (1-\delta) = \frac{Y_t}{K_{t-1}}\left(1 - \frac{C_t}{Y_t} - \frac{G_t}{Y_t}\right).$$

This expression is convenient because we know the steady-state values of K_t/K_{t-1}, Y_t/K_{t-1}, C_t/Y_t, and G_t/Y_t:

- K grows at rate γ in the steady state, so the steady-state value of K_t/K_{t-1} is $e^{\gamma} \cong 1 + \gamma$,
- We calibrate the model to a chosen steady-state value of G_t/Y_t,
- You will show as part of the project that the steady-state value of Y_t/K_{t-1} is $(\rho + \gamma + \delta)/\alpha$,
- You will likewise show that the steady-state value of C_t/Y_t is $1 - (G/Y)^* - \alpha(\gamma + \delta)/(\rho + \gamma + \delta)$.

Taking logs of both sides of this equation,

$$\ln\left[\exp(\Delta k_t) - (1-\delta)\right] = y_t - k_{t-1} + \ln\left[1 - \exp(c_t - y_t) - \exp(g_t - y_t)\right]. \qquad (15)$$

Two parts of equation (15) warrant clarification. First, on the left-hand side,

$$\ln\left(\frac{K_t}{K_{t-1}}\right) = \ln K_t - \ln K_{t-1} = k_t - k_{t-1} \equiv \Delta k_t,$$

so $K_t / K_{t-1} = \exp(\Delta k_t)$. Second, on the right-hand side, $\ln(C_t / Y_t) = c_t - y_t$, so $C_t / Y_t = \exp(c_t - y_t)$, and similarly for G.

Equation (15) expresses equation (2) in terms of the logs of the variables, but it is highly nonlinear. We therefore approximate the nonlinear parts of (15) with first-order Taylor series, first for the function on the left of the equal sign then for the one on the right.

Working first with the left-hand side, define the function

$$f_1(\Delta k_t) \equiv \ln\left[\exp(\Delta k_t) - (1-\delta)\right].$$

We can approximate $f_1(\Delta k_t)$ in a neighborhood around the steady state by

$$f_1(\Delta k_t) \cong f_1(\gamma) + f_1'(\gamma)(\Delta k_t - \gamma),$$

where the growth rate γ is known to be the steady-state value of Δk_t. Because we are going to express the model in terms of *deviations* from the steady state, we need worry only about the second term. Using the chain rule,

$$f_1'(\Delta k_t) = \frac{1}{\exp(\Delta k_t) - (1-\delta)} \exp(\Delta k_t).$$

Evaluating this expression at the steady-state value $\Delta k_t = \gamma$ and taking advantage of the approximation $\exp(\gamma) \cong 1 + \gamma$ gives

$$f_1'(\gamma) \cong \frac{1+\gamma}{(1+\gamma)-(1-\delta)} = \frac{1+\gamma}{\delta+\gamma}. \tag{16}$$

Turning to the right-hand side of (15), we now tackle the nonlinear function

$$f_2\left[(c_t - y_t), (g_t - y_t)\right] \equiv \ln\left[1 - \exp(c_t - y_t) - \exp(g_t - y_t)\right].$$

For this function of two variables, the first-order Taylor-series approximation is

$$
\begin{aligned}
f_2\left[(c_t - y_t),(g_t - y_t)\right] \cong\ & f_2\left[(c-y)^*,(g-y)^*\right] \\
& + \frac{\partial f_2}{\partial(c_t - y_t)}\left[(c-y)^*,(g-y)^*\right]\cdot\left[(c_t - y_t) - (c-y)^*\right] \\
& + \frac{\partial f_2}{\partial(g_t - y_t)}\left[(c-y)^*,(g-y)^*\right]\cdot\left[(g_t - y_t) - (g-y)^*\right],
\end{aligned}
$$

where $(c - y)^*$ and $(g - y)^*$ are the steady-state values. The steady-state value of $(g - y)^*$ is the log of the calibration parameter $(G/Y)^*$. From the expression in the bullets above,

$$
\left(c^* - y^*\right) = \ln\left(\frac{C}{Y}\right)^* = \ln\left\{\left[1 - \left(\frac{G}{Y}\right)^*\right] - \alpha\frac{\gamma+\delta}{\rho+\gamma+\delta}\right\}
$$

As before, our interest in deviations around the steady state focuses our attention on the last two terms of the Taylor approximation. Computing the partial derivative with respect to $(c_t - y_t)$,

$$
\begin{aligned}
\frac{\partial f_2}{\partial(c_t - y_t)}\left[(c_t - y_t),(g_t - y_t)\right] &= \frac{1}{1 - \exp(c_t - y_t) - \exp(g_t - y_t)}\cdot\left(-\exp(c_t - y_t)\right) \\
&= -\frac{(C_t/Y_t)}{1 - (C_t/Y_t) - (G_t/Y_t)}.
\end{aligned}
$$

Evaluating this at the steady-state values gives

$$
\begin{aligned}
\frac{\partial f_2}{\partial(c_t - y_t)}\left[(c-y)^*,(g-y)^*\right] &= -\frac{(C/Y)^*}{1 - (C/Y)^* - (G/Y)^*} \\
&= -\frac{\left[1 - (G/Y)^*\right] - \alpha\dfrac{\gamma+\delta}{\rho+\gamma+\delta}}{1 - \left[\left[1 - (G/Y)^*\right] - \alpha\dfrac{\gamma+\delta}{\rho+\gamma+\delta}\right] - (G/Y)^*} \\
&= 1 - \frac{1 - (G/Y)^*}{\alpha\dfrac{\gamma+\delta}{\rho+\gamma+\delta}} \equiv 1 - \frac{\rho+\gamma+\delta}{\alpha(\gamma+\delta)}\left[1 - (G/Y)^*\right].
\end{aligned}
$$

Since $(G/Y)^*$ is a calibrated parameter, this partial derivative can be evaluated numerically at the steady state in terms of the parameters of the model. Following similar logic, it is straightforward to show that

$$\frac{\partial f_2}{\partial(g_t - y_t)}\left[(c-y)^*,(g-y)^*\right] = -\frac{\rho+\gamma+\delta}{\alpha(\gamma+\delta)}(G/Y)^*.$$

We are now prepared to express equation (15) in terms of deviations around the steady state. Subtracting the steady-state values yields

$$\begin{aligned}\ln\left[\exp(\Delta k_t)-(1-\delta)\right]-\ln\left[\exp(\gamma)-(1-\delta)\right] &= \left(y_t - y_t^*\right)-\left(k_{t-1}-k_{t-1}^*\right)\\ &+\ln\left[1-\exp(c_t-y_t)-\exp(g_t-y_t)\right]-\ln\left[1-(C/Y)^*-(G/Y)^*\right]\end{aligned}$$

or

$$\begin{aligned}f_1(\Delta k_t)-f_1(\gamma) &= \left(y_t-y_t^*\right)-\left(k_{t-1}-k_{t-1}^*\right)\\ &+f_2\left[(c_t-y_t),(g_t-y_t)\right]-f_2\left[(c-y)^*,(g-y)^*\right].\end{aligned}$$

Substituting from the Taylor-series approximations for the expression on the left and the expression on the second line of the right:

$$\begin{aligned}\frac{1+\gamma}{\delta+\gamma}(\Delta k_t-\gamma) &= \left(y_t-y_t^*\right)-\left(k_{t-1}-k_{t-1}^*\right)\\ &+\left(1-\frac{\rho+\gamma+\delta}{\alpha(\gamma+\delta)}\left[1-(G/Y)^*\right]\right)\left[\left(c_t-c_t^*\right)-\left(y_t-y_t^*\right)\right]\\ &+\left(-\frac{\rho+\gamma+\delta}{\alpha(\gamma+\delta)}(G/Y)^*\right)\left[\left(g_t-g_t^*\right)-\left(y_t-y_t^*\right)\right].\end{aligned}$$

Or, in terms of the deviations,

$$\begin{aligned}\frac{1+\gamma}{\delta+\gamma}\left(\tilde{k}_t-\tilde{k}_{t-1}\right) &= \left(\tilde{y}_t-\tilde{k}_{t-1}\right)\\ &+\left(1-\frac{\rho+\gamma+\delta}{\alpha(\gamma+\delta)}\left[1-(G/Y)^*\right]\right)(\tilde{c}_t-\tilde{y}_t)\\ &-\left(\frac{\rho+\gamma+\delta}{\alpha(\gamma+\delta)}(G/Y)^*\right)(\tilde{g}_t-\tilde{y}_t).\end{aligned}$$

Collecting terms in $\tilde{y}_t$ and $\tilde{k}_{t-1}$ simplifies this to

$$\frac{1+\gamma}{\delta+\gamma}\tilde{k}_t = \frac{\rho+\gamma+\delta}{\alpha(\gamma+\delta)}\tilde{y}_t + \frac{1-\delta}{\delta+\gamma}\tilde{k}_{t-1}$$
$$+\left(1-\frac{\rho+\gamma+\delta}{\alpha(\gamma+\delta)}\left[1-(G/Y)^*\right]\right)\tilde{c}_t$$
$$-\frac{\rho+\gamma+\delta}{\alpha(\gamma+\delta)}(G/Y)^*\tilde{g}_t,$$

or

$$\begin{aligned}(1+\gamma)\tilde{k}_t &= (1-\delta)\tilde{k}_{t-1} + \frac{\rho+\gamma+\delta}{\alpha}\tilde{y}_t \\ &+\left[(\gamma+\delta)-\frac{(\rho+\gamma+\delta)\left[1-(G/Y)^*\right]}{\alpha}\right]\tilde{c}_t \\ &-\frac{\rho+\gamma+\delta}{\alpha}\left(\frac{G}{Y}\right)^*\tilde{g}_t.\end{aligned} \tag{17}$$

Equation (17) is an approximation of equation (2) that is linear in terms of the log-deviations from the steady state, which is what we require in order to solve the model. Note again that all of the coefficients appearing in (17)—γ, δ, λ, ρ, and $(G/Y)^*$ —are parameters that we assume to be known.

Leaving equations (3) and (4) for you to examine as part of the exercise, we now turn to the log-linearization of equation (5), which needs special attention because of the expectation operator on the right-hand side. If we take logs of both sides, we get

$$-c_t = -\rho + \ln\left[E_t\left(R_{t+1}/C_{t+1}\right)\right]. \tag{18}$$

If we could just take the expectation operator outside the log operator, this could easily be reduced to an expression in $\tilde{c}_t$, $\tilde{r}_{t+1}$, and $\tilde{c}_{t+1}$. However, the log of an expectation is *not* equal to the expectation of the log, so it is not so simple.

In order to simplify (18) we must make a simple assumption about the distribution of the random shocks. If the ε shocks both follow the normal probability distribution, then all of the log variables in the linearized system will also be normal random variables and the capital-letter variables will follow the *log-normal* distribution. If R and C are distributed log-normally, then

$$\ln\left[E_t\left(\frac{R_{t+1}}{C_{t+1}}\right)\right] = E_t\left(r_{t+1}-c_{t+1}\right)+\zeta,$$

where ζ (zeta) is a constant. Because ζ is the same in the steady state as outside, it will cancel when we take deviations from the steady state, so we need not evaluate it.

Applying this convenient solution to the expectation problem and taking deviations from the steady state in (18) yields

$$-\tilde{c}_t = E_t \tilde{r}_{t+1} - E_t \tilde{c}_{t+1}, \tag{19}$$

which is our log-deviation version of equation (5).

E. Calibration vs. Estimation in Empirical Economics

Early real-business-cycle models have used a different method of empirical validation than was typical of earlier macroeconomic models. The traditional approach is to specify a model consisting of a set of equations having unknown parameters to be estimated, then use econometric techniques to estimate the parameters and test whether their signs and magnitudes are consistent with the model's assumptions. The strength of this approach is that it provides formal statistical tests of the hypotheses that underlie the model. Its weakness is that restrictive assumptions called ***identifying restrictions*** have to be made in order to estimate the model. Identifying restrictions include assumptions about what variables are exogenous rather than endogenous (recall that exogenous variables are assumed to be unaffected by any other variables in the model) and which variables can be excluded from having a direct effect in the equation determining each other variable. If an estimated model is based on inappropriate identifying restrictions, the estimates and hypothesis tests will generally be invalid. Unfortunately, macroeconomists rarely have great confidence about which variables are exogenous and about which variables can be safely omitted from each equation. As a result, the validity of most econometric results is open to challenge by those who believe that inappropriate restrictions were assumed.

RBC modelers have often eschewed econometric estimation in favor of a technique called ***calibration and simulation***. This technique is somewhat familiar to you from the simple empirical analysis of the Solow and human-capital models. Recall that in order to assess the empirical validity of those models we posited values for such parameters as capital's share of GDP and the depreciation rate by rather casual empirical observation. We then plugged in these values to calculate the model's predictions about properties such as the speed of convergence and compared them to actual observed rates of convergence.

The procedure for RBC models is similar, but it is more complicated because the RBC models are stochastic rather than deterministic. In order to calibrate an RBC model, one must obtain external estimates of the parameters of the production function and utility function, as well as other parameters such as the rate of growth of the population. Using these estimates, the model is solved repeatedly using different sequences of randomly generated values for the random variables of the model—the disturbances to the rate of technological progress, ε_A. From the results of these many replicated simulations, basic properties of the simulated variables are computed, such as relative variances of the variables, correlations between them, and the time pattern of co-movements between variables. If the calculated properties of the simulated model correspond to those of actual macroeconomic data, then the validity of the model is supported. If some properties differ, then the model is deemed unable to explain these aspects of economic cycles.

Some of the parameters of RBC models are ones that would be very difficult to estimate econometrically but can be approximated within reasonable bounds by common sense. For example, it would be difficult or impossible to design an econometric equation that would allow the rate of time preference to be directly estimated. However, we know that plausible values for this parameter must be in the same range as real interest rates, conservatively between zero and ten percent per year.

The advantage of the calibration approach, which is emphasized by those who favor RBC models, is that it avoids the awkward and questionable identifying restrictions that would be necessary if we were to attempt to estimate these difficult parameters. However, the calibration approach has disadvantages as well. When there is no consensus about the value of a parameter, one must simply guess at its value in a calibrated model. Moreover, calibrated models do not provide standard errors or test statistics for the parameters since the parameters are not estimated. The statistical properties of simulation results in calibrated models are just being developed; most simulation results in published studies are devoid of statistical tests.

Much progress has been made in the last decade allowing estimation of parameters in simulated models. In particular, "Bayesian" statistical methods have become the norm in simulating business-cycle models. In Bayesian models, one specifies a "prior" probability distribution that specifies our *a priori* beliefs or knowledge about the parameter. The Bayesian algorithm then combines our prior distribution with the information provided by the sample data to calculate a "posterior" probability distribution for the parameter. The model is simulated using the posterior distributions for all parameters. Often thousands of replications are done with parameter values being drawn randomly from the posterior distributions rather than simply set at their mean values. A useful discussion of the Bayesian approach to simulated macroeconomic models can be found in Fernández-Villaverde and Rubio-Ramírez (2004).

F. Suggestions for Further Reading

Note: Empirical tests of RBC models are discussed in detail in Chapter 13 of the coursebook.

Seminal RBC papers

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