# Economics 312 — Spring 2014
# Daily Problem #27 — March 28

*Note: Appendices 9B and 9C of HGL discusses these questions.*

One of the most common time-series processes is the first-order autoregressive process: AR(1). Suppose that the error term of a time-series regression follows an AR(1) process: $(1-\rho L)u_t = \varepsilon_t$, or $u_t = \rho u_{t-1} + \varepsilon_t$, where $-1 < \rho < 1$ and $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$ is serially uncorrelated white noise.

1. Using the result of Daily Problem #25, show that we can write $u_t = \sum_{s=0}^{\infty} \rho^s \varepsilon_{t-s}$.

2. Using the property that $\operatorname{cov}(\varepsilon_t, \varepsilon_{t-s}) = 0$ for all $s \neq 0$, use the expression in question 1 to show that $\operatorname{var}(u_t) \equiv \sigma_u^2 = \sigma_\varepsilon^2 \sum_{s=0}^{\infty} (\rho^2)^s = \frac{\sigma_\varepsilon^2}{1-\rho^2}$. (Hint: Use the equation $u_t = \rho u_{t-1} + \varepsilon_t$, take the variance of both sides, and note that $\operatorname{var}(u_t) = \sigma_u^2$ for all *t* and that $\varepsilon_t$ is uncorrelated with anything that happened before *t*.)

3. Show that $\operatorname{cov}(u_t, u_{t-1}) = \rho\sigma_u^2$ and that, more generally, $\operatorname{cov}(u_t, u_{t-s}) = \rho^s \sigma_u^2$. (You can proceed most easily in a manner similar to the hint above.)

4. Using these results, show that the covariance matrix of the error vector $\mathbf{u} \equiv \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_T \end{bmatrix}$ is

$$E(\mathbf{u}\mathbf{u}') = \sigma_u^2 \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{T-1} \\ \rho & 1 & \rho & & \vdots \\ \rho^2 & \rho & 1 & & \rho^2 \\ \vdots & & & \ddots & \rho \\ \rho^{T-1} & \cdots & \rho^2 & \rho & 1 \end{bmatrix}.$$

5. *(Optional bonus question)* Show that if $\mathbf{P} = \begin{bmatrix} \frac{1}{\sqrt{1-\rho^2}} & 0 & 0 & \cdots & 0 \\ -\rho & 1 & 0 & \cdots & 0 \\ 0 & -\rho & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & -\rho & 1 \end{bmatrix}$, then the regression $\mathbf{Py} = \mathbf{PX} + \mathbf{Pe}$ corresponds to the GLS estimator of Appendix 9C, and that $E\left[\mathbf{Pe}(\mathbf{Pe})'\right] = E[\mathbf{Pee}'\mathbf{P}'] = \mathbf{P}E(\mathbf{ee}')\mathbf{P}' = \sigma_v^2\mathbf{I}_T$ so that the transformed GLS model satisfies the standard MR assumptions.